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On canonical bending relationships for plates

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Abstract

In recent years, a series of papers have appeared on algebraic relationships between the solutions (e.g., deflections, buckling loads and frequencies) of a given higher-order plate theory and the classical plate theory. The bending relationships, for example, can be used to generate the transverse deflection of a plate according to the particular higher-order theory from the known deflection of the same problem according to the classical plate theory. In the present study relationships between the bending solutions of several higher-order plate theories and the classical plate theory are obtained in a canonical form (i.e., one set of relationships contain several theories and they can be specialized to a specific theory by assigning values to the constants appearing in the relationships). Numerical examples of bending solutions for rectangular plates with various boundary conditions are presented to show how the relations can be used to determine the deflections and bending moments for various theories. The relationships are validated by comparing the numerical results obtained using the relationships for the Mindlin plate theory against those computed using the ABAQUS finite element program.

Keywords: Bending relationships; Canonical form; Mindlin plate theory; Reissner plate theory; Levinson plate theory

1. Introduction

Classical plate theory, also known as the *Kirchhoff* plate theory (Szilard, 1974; Reddy, 1999), is used extensively to analyze plates whose length-to-thickness ratio is of the order of 25 or greater (i.e., thin plates). Bending, buckling and vibration solutions of plates according to the classical plate theory have been well-documented in many standard textbooks like Timoshenko and Woinowsky-Krieger (1959), Mansfield (1989) and Reddy (1999). When the plate's properties are anisotropic and the length-to-thickness ratio is less than 25, the effect of transverse shear deformation on deflections, buckling loads and frequencies can be significant, and it is necessary to use theories that account for transverse shear deformation.

There exists a number of plate theories that account for transverse shear strains and stresses and provide various degrees of refinement to the classical plate theory. The more commonly known refined plate theories are the *Reissner* plate theory (Reissner, 1945, 1947), the *Mindlin* plate theory (Mindlin, 1951) and the

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Levinson plate theory (Levinson, 1980). Except for the Reissner plate theory, the aforementioned plate theories are developed using an assumed displacement field. The Reissner theory, on the other hand, is based on an assumed stress field. The Mindlin plate theory is often referred to as a first-order [1] shear deformation plate theory while the Levinson plate theory is a third-order theory. While shear deformable plate theories more accurately describe the kinematic behavior of thick plates than the classical plate theory, analytical solutions of shear deformation theories are more difficult to formulate. A limited number of analytical solutions of the first-order shear deformation plate theory may be found in the textbooks by Timoshenko and Woinowsky-Krieger (1959) and Reddy (1999, 2002); analytical results to selected plate problems using the Levinson plate theory were developed by Levinson and Cooke (1983) and Cooke and Levinson (1983), whereas Salerno and Goldberg (1960) provided analytical solutions of the Reissner plate theory.

The amount of literature available on analytical solutions of refined theories is hence limited and it is useful to have relationships that connect solutions of the classical plate theory to those of the available refined theories so that one can immediately determine the solution of a plate problem according to a shear deformation theory from the corresponding classical plate theory solution. The relationships can also reveal the effect of shear deformation in an explicit manner.

In recent years Wang and his co-researchers (see Wang et al., 2001 and references therein) have developed relationships connecting the solutions of shear deformation plate theories to those of the classical plate theory. As the thin plate solutions are available in textbooks, these relationships provide an efficient and quick way to determine the solutions based on shear deformation theories. However, so far the bending relationships have been specifically determined for various plate shapes (circular, rectangular, sectorial, annular sectorial and polygonal), boundary conditions (Navier- and Levy-types of boundary conditions) and different shear deformable plate theories (Mindlin, Reissner and Levinson plate theories). Since the existing relationships are valid for a particular plate problem and a particular theory (see Wang et al., 1999, 2001; Reddy et al., 2001; Lee et al., 2002), this poses an undesirable computation inefficiency if one were to compare the solutions provided by various shear deformable plate theories.

The present study overcomes the above shortcoming by furnishing the bending relationships in a canonical form with constants that can be specialized readily for the different plate theories. Existing bending relationships are shown to be special cases of those presented here. Additional analytical solutions of plate problems previously not reported through relationships, such as simply supported plates with edge moments and clamped plates with transverse loads are solved in this study and the numerical results are verified with existing analytical results or with those computed using the commercial finite element software, ABAQUS (2001). To aid the use of bending relationships for readers, specialized relationships developed for the plate problems treated in this study have been compiled in Appendix A, with the corresponding thin plate solutions given in Appendix B.

2. Governing equations

Consider the bending of an isotropic and homogeneous plate subjected to a transverse load $q(x, y)$. The governing equations of equilibrium according to the classical plate theory, the first-order shear deformation theory, the Reissner plate theory (see Panc, 1975; Reddy, 1999) and the Levinson plate theory (1980) are given by

[1] The order of a plate theory refers to the order of the transverse coordinate z in the expansion of the displacement field.

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q(x,y), \tag{2.1a}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x, \tag{2.1b}$$

$$\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} = Q_y, \tag{2.1c}$$

where Q_x and Q_y are the transverse shear forces per unit length while M_{xx}, M_{yy} and M_{xy} the moments per unit length.

Levinson (1980) has adopted the above set of equilibrium equations in his formulation of a third-order plate theory but the equations are variationally not consistent with the assumed displacement field. Therefore the Levinson plate theory has been perceived by many as being *variational inconsistent* and has prompted several discussion on its correctness in the open literature (Levinson, 1987; Hutchinson, 1987; Rychter, 1987). A variational consistent third-order shear deformation plate theory has been developed by Reddy (1984). It requires the introduction of higher-order stress resultants that are physically difficult to interpret.

3. Shear deformation plate theories

Although the equilibrium equations in Eqs. (2.1) are valid for various theories, the stress resultants (M_{xx}, M_{yy}, M_{xy}, Q_x and Q_y) in each theory are related to the generalized displacements differently. One can however express the stress resultants for the Mindlin, Reissner and Levinson plate theories in terms of the displacements in the following canonical form:

$$M_{xx}^H = D\left[\mathscr{A}\left(\frac{\partial \phi_x^H}{\partial x} + v\frac{\partial \phi_y^H}{\partial y}\right) - (1-\mathscr{A})\left(\frac{\partial^2 w^H}{\partial x^2} + v\frac{\partial^2 w^H}{\partial y^2}\right)\right] + \frac{\mathscr{B}q}{1-v}, \tag{3.1a}$$

$$M_{yy}^H = D\left[\mathscr{A}\left(\frac{\partial \phi_y^H}{\partial y} + v\frac{\partial \phi_x^H}{\partial x}\right) - (1-\mathscr{A})\left(\frac{\partial^2 w^H}{\partial y^2} + v\frac{\partial^2 w^H}{\partial x^2}\right)\right] + \frac{\mathscr{B}q}{1-v}, \tag{3.1b}$$

$$M_{xy}^H = \frac{1}{2}D(1-v)\left[\mathscr{A}\left(\frac{\partial \phi_x^H}{\partial y} + \frac{\partial \phi_y^H}{\partial x}\right) - 2(1-\mathscr{A})\left(\frac{\partial^2 w^H}{\partial x \partial y}\right)\right], \tag{3.1c}$$

$$Q_x^H = \mathscr{A}K_s Gh\left(\phi_x^H + \frac{\partial w^H}{\partial x}\right), \tag{3.1d}$$

$$Q_y^H = \mathscr{A}K_s Gh\left(\phi_y^H + \frac{\partial w^H}{\partial y}\right), \tag{3.1e}$$

$$\mathscr{M}^H = D\left[\mathscr{A}\left(\frac{\partial \phi_x^H}{\partial x} + \frac{\partial \phi_y^H}{\partial y}\right) - (1-\mathscr{A})\nabla^2 w^H\right] + \frac{2\mathscr{B}q}{1-v^2}, \tag{3.1f}$$

where the superscript H denotes the shear deformable plate quantities, h is the plate thickness, D is the plate flexural rigidity, G is the shear modulus and v is the Poisson ratio. The variables w, ϕ_x and ϕ_y in

Eqs. (3.1a–f) are the kinematic displacements assumed for bending by the various plate theories which will be characterized separately below and $\mathscr{M}$ $(= (M_{xx} + M_{yy})/(1 + \nu))$ is the Marcus moment or moment sum (Marcus, 1932). The shear coefficient K_s and the constants $\mathscr{A}$ and $\mathscr{B}$ will be defined for the respective plate theories as follows.

The Mindlin plate theory. In the Mindlin plate theory, w is taken to be the mid-plane transverse displacement while ϕ_x and ϕ_y represent the normal rotations about the y- and x-axes, respectively. The parameters $\mathscr{A}$, $\mathscr{B}$ and K_s have the values

$$\mathscr{A} = 1, \quad \mathscr{B} = 0, \quad K_s = \kappa^2, \tag{3.2}$$

where κ^2 is the Mindlin shear correction factor that is dependent on the plate loading, geometrical and boundary conditions. Generally for isotropic plates, it has commonly been taken as 5/6.

The Reissner plate theory. Unlike the Mindlin and Levinson plate theories, the Reissner plate theory is derived through a priori assumption of stress distributions and one obtains the displacement kinematics, via matching the work done of the stress through the plate thickness with that of an equivalent force (Reissner, 1945, 1947). As a result, w is the *weighted average* transverse plate deflection while ϕ_x and ϕ_y are the *equivalent* normal rotations about the y- and x-axes, respectively. For the theory, the parameters $\mathscr{A}$, $\mathscr{B}$ and K_s take on the values of

$$\mathscr{A} = 1, \quad \mathscr{B} = \frac{h^2 \nu}{10}, \quad K_s = \frac{5}{6}. \tag{3.3}$$

The Levinson plate theory. The displacement field hypothesized by Levinson in his formulation belongs to that of a third-order plate theory while maintaining the transverse inextensibility; here, it is to be remarked that all the theories concerned in this study, including the Kirchhoff plate theory, retains such a hypothesis. With a cubic polynomial representation of the transverse coordinate in the displacement field, the normal to the mid-plane is allowed to warp during deformation in a non-linear fashion with zero rotation at both free surfaces. Therefore, ϕ_x and ϕ_y are the rotations of the *warped normals* about the y- and x-axes, respectively, while w denotes the mid-plane displacement (Levinson, 1987). The parameters $\mathscr{A}$, $\mathscr{B}$ and K_s for the Levinson plate theory become

$$\mathscr{A} = \frac{4}{5}, \quad \mathscr{B} = 0, \quad K_s = \frac{5}{6}. \tag{3.4}$$

Using Eqs. (3.1a–c) and (3.1f), the shear forces Q_x^H and Q_y^H can also be determined from the equilibrium equations, Eqs. (2.1b) and (2.1c), respectively, in terms of the moment sum as

$$Q_x^H = \frac{\partial \mathscr{M}^H}{\partial x} + \frac{1}{2} D(1-\nu) \left[\mathscr{A} \frac{\partial}{\partial y} \left(\frac{\partial \phi_x^H}{\partial y} - \frac{\partial \phi_y^H}{\partial x} \right) \right] - \frac{\mathscr{B}}{1+\nu} \frac{\partial q}{\partial x}, \tag{3.5a}$$

$$Q_y^H = \frac{\partial \mathscr{M}^H}{\partial y} - \frac{1}{2} D(1-\nu) \left[\mathscr{A} \frac{\partial}{\partial x} \left(\frac{\partial \phi_x^H}{\partial y} - \frac{\partial \phi_y^H}{\partial x} \right) \right] - \frac{\mathscr{B}}{1+\nu} \frac{\partial q}{\partial y}. \tag{3.5b}$$

Using Eqs. (3.5a,b) and (2.1a), one can now establish the first governing equation for the bending of thick plates as

$$\nabla^2 \left(\mathscr{M}^H - \frac{\mathscr{B}}{1+\nu} q \right) = -q. \tag{3.6}$$

Similarly, from Eqs. (3.1d), (3.1e), (3.1f) and (2.1a), the second thick plate governing equation is written as

$$K_sGh\left(\nabla^2 w^H + \frac{\mathscr{M}^H}{D}\right) = -\left(1 - \frac{\mathscr{B}c^2}{1+v}\right)q, \tag{3.7}$$

where

$$c^2 = \frac{2K_sGh}{D(1-v)} = \frac{12K_s}{h^2}.$$

By differentiating Eqs. (3.5a) and (3.5b) with respect to y and x, respectively, and combining with Eqs. (3.1d) and (3.1e) to eliminate the transverse deflection w^H, one arrives at the last essential governing equation for thick plates

$$\nabla^2\left(\frac{\partial\phi_x^H}{\partial y} - \frac{\partial\phi_y^H}{\partial x}\right) = c^2\left(\frac{\partial\phi_x^H}{\partial y} - \frac{\partial\phi_y^H}{\partial x}\right). \tag{3.8}$$

With a sixth-order system of governing differential equations for the concerned shear deformable plate theories, it will be necessary to have a set of six boundary conditions for unique solutions. Hence along a plate boundary, the conditions to be imposed for each support type are

- simply supported edge:

$$w^H = M_{nn}^H = \phi_s^H = 0, \tag{3.9}$$

- clamped edge:

$$w^H = \phi_n^H = \phi_s^H = 0, \tag{3.10}$$

- free edge:

$$M_{nn}^H = M_{ns}^H = Q_n^H = 0, \tag{3.11}$$

where n is the outward normal direction to the plate boundary and s the tangential direction.

4. Kirchhoff plate theory

For the bending of thin plates, the Kirchhoff stress resultant–displacement relationships are given as follows:

$$M_{xx}^K = -D\left(\frac{\partial^2 w^K}{\partial x^2} + v\frac{\partial^2 w^K}{\partial y^2}\right), \tag{4.1a}$$

$$M_{yy}^K = -D\left(\frac{\partial^2 w^K}{\partial y^2} + v\frac{\partial^2 w^K}{\partial x^2}\right), \tag{4.1b}$$

$$M_{xy}^K = -D(1-v)\left(\frac{\partial^2 w^K}{\partial x \partial y}\right), \tag{4.1c}$$

where the superscript K denotes the Kirchhoff plate quantities. As known, the normality assumption made in the Kirchhoff plate theory will result in zero transverse shear strains, thereby ignoring the transverse shear deformation. As a result, the Kirchhoff shear forces computed by the constitutive relations will be zero. Thus to compute shear forces for thin plates, one will have to resort to the conditions of equilibrium, Eqs. (2.1b) and (2.1c) to give

$$Q_x^K = -D\frac{\partial}{\partial x}(\nabla^2 w^K), \quad Q_y^K = -D\frac{\partial}{\partial y}(\nabla^2 w^K). \tag{4.2a, b}$$

Introducing the moment sum $\mathscr{M}$, the expressions for the Kirchhoff shear forces can be simplified to

$$Q_x^K = \frac{\partial \mathscr{M}^K}{\partial x}, \quad Q_y^K = \frac{\partial \mathscr{M}^K}{\partial y}, \tag{4.3a, b}$$

where $\mathscr{M}^K = -D\nabla^2 w^K$. Note that the Kirchhoff stress resultants in Eqs. (4.1)–(4.3) can also be obtained from Eqs. (3.1) and (3.5) by setting $\mathscr{A} = \mathscr{B} = 0$.

The governing equation for the bending of thin plates can be established by substituting Eqs. (4.3a) and (4.3b) into Eq. (2.1a)

$$\nabla^2 \mathscr{M}^K = -q \quad \Rightarrow \quad D\nabla^4 w^K = q. \tag{4.4a, b}$$

From Eq. (4.4b), one can see that the governing differential equation of the classical thin plate theory is of fourth order and will hence require a total of four independent boundary conditions for solutions. As such, the Kirchhoff boundary conditions to be specified for each support type along a plate edge are defined as follows:

- simply supported edge:

$$w^K = M_{nn}^K = 0, \tag{4.5}$$

- clamped edge:

$$w^K = \frac{\partial w^K}{\partial n} = 0, \tag{4.6}$$

- free edge:

$$M_{nn}^K = V_n^K = 0, \tag{4.7}$$

where V_n^K $(= Q_n^K + (\partial M_{ns}^K/\partial s))$ is the *effective* shear force. As mentioned by Reissner (1985), the number of Kirchhoff boundary conditions especially along a free plate edge poses a *boundary paradox*, as opposed to the common engineering knowledge that there should be three natural boundary conditions along such a plate edge, i.e., vanishing bending and twisting moments, as well as shear force. To remove such a paradox for this case, a contraction of two boundary conditions into one has been made to introduce the effective shear force using the variational approach.

5. Relationships between shear deformation and classical theories

Based on the order of the governing equation(s) of the plate theories, it is evident that it is comparatively easier to seek solutions using the thin plate theory. Also owing to the long existence of the Kirchhoff plate theory, thin plate solutions have been well-documented in many standard texts on plate problems like Timoshenko and Woinowsky-Krieger (1959), Szilard (1974), Panc (1975), Mansfield (1989) and Reddy (1999). As much as it is important to model moderately thick plates more accurately using the shear deformable plate theories, it has been mathematically difficult to develop thick plate solutions analytically. Hence, to have bending relationships that can predict thick plate solutions using the corresponding thin plate results provides an attractive option as these bending relationships will also help engineers and researchers to elucidate the significance and effect of transverse shear deformation.

This section will illustrate the procedures how the solutions of a shear deformable plate theory can be related to the results of the thin plate theory. First by the analogy of load equivalence, one may express a relationship for the moment sum by setting Eq. (3.6) equal to (4.4a) and then solving the Laplace equation,

$$\nabla^2\left(\mathscr{M}^H - \frac{\mathscr{B}}{1+v}q\right) = \nabla^2\mathscr{M}^K \quad \Rightarrow \quad \mathscr{M}^H = \mathscr{M}^K + \frac{\mathscr{B}}{1+v}q + D\nabla^2\Phi^H, \tag{5.1a, b}$$

where Φ^H is an intrinsic plate function that is strictly bi-harmonic, i.e.,

$$\nabla^4\Phi^H = 0. \tag{5.2}$$

Substituting Eqs. (4.4a) and (5.1b) into Eq. (3.7) will furnish the Kirchhoff-shear-deformable deflection relationship which is

$$w^H = w^K + \frac{1}{K_s Gh}\left(1 - \frac{\mathscr{B}c^2}{2}\right)\mathscr{M}^K - \Phi^H + \Psi^H, \tag{5.3}$$

where Ψ^H is another intrinsic plate function that will satisfy the Laplace equation

$$\nabla^2\Psi^H = 0. \tag{5.4}$$

For the differential equation given in Eq. (3.8), one can express the solution as Ω^H that will constitute the last of the three intrinsic plate functions for all the bending relationships

$$\nabla^2\Omega^H = c^2\Omega^H, \quad \Omega^H = \frac{\partial\phi_x^H}{\partial y} - \frac{\partial\phi_y^H}{\partial x}. \tag{5.5a, b}$$

In view of Eqs. (3.1d), (3.1e), (3.5a), (3.5b), (4.3a), (4.3b), (5.1b), (5.3) and (5.5b) and by a series of algebraic manipulation, the rotation–slope relationships are determined as

$$\phi_x^H = -\frac{\partial w^K}{\partial x} - \frac{1}{K_s Gh}\left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)Q_x^K + \frac{\partial}{\partial x}\left(\frac{\mathscr{D}}{\mathscr{A}}\nabla^2\Phi^H + \Phi^H - \Psi^H\right) + \frac{1}{c^2}\frac{\partial\Omega^H}{\partial y}, \tag{5.6a}$$

$$\phi_y^H = -\frac{\partial w^K}{\partial y} - \frac{1}{K_s Gh}\left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)Q_y^K + \frac{\partial}{\partial y}\left(\frac{\mathscr{D}}{\mathscr{A}}\nabla^2\Phi^H + \Phi^H - \Psi^H\right) - \frac{1}{c^2}\frac{\partial\Omega^H}{\partial x}, \tag{5.6b}$$

where

$$\mathscr{D} = \frac{D}{K_s Gh}. \tag{5.6c}$$

Finally for the stress resultant relationships, one can substitute Eqs. (5.3), (5.6a) and (5.6b) into Eqs. (3.1a–e) to write

$$M_{xx}^H = M_{xx}^K - \mathscr{B}\frac{\partial Q_y^K}{\partial y} - D(1-v)\frac{\partial\Lambda^-}{\partial y} + D\nabla^2\Phi^H, \tag{5.7a}$$

$$M_{yy}^H = M_{yy}^K - \mathscr{B}\frac{\partial Q_x^K}{\partial x} - D(1-v)\frac{\partial\Lambda^+}{\partial x} + D\nabla^2\Phi^H, \tag{5.7b}$$

$$M_{xy}^H = M_{xy}^K + \mathscr{B}\frac{\partial Q_y^K}{\partial x} + \frac{1}{2}D(1-v)\left(\frac{\partial\Lambda^+}{\partial y} + \frac{\partial\Lambda^-}{\partial x}\right), \tag{5.8}$$

$$Q_x^H = Q_x^K + D\frac{\partial}{\partial x}(\nabla^2 \Phi^H) + \frac{1}{2}D(1-\nu)\left(\mathscr{A}\frac{\partial \Omega^H}{\partial y}\right), \tag{5.9a}$$

$$Q_y^H = Q_y^K + D\frac{\partial}{\partial y}(\nabla^2 \Phi^H) - \frac{1}{2}D(1-\nu)\left(\mathscr{A}\frac{\partial \Omega^H}{\partial x}\right), \tag{5.9b}$$

where

$$\Lambda^+ = \frac{\partial}{\partial x}(\mathscr{D}\nabla^2\Phi^H + \Phi^H - \Psi^H) + \frac{\mathscr{A}}{c^2}\frac{\partial \Omega^H}{\partial y}, \tag{5.10a}$$

$$\Lambda^- = \frac{\partial}{\partial y}(\mathscr{D}\nabla^2\Phi^H + \Phi^H - \Psi^H) - \frac{\mathscr{A}}{c^2}\frac{\partial \Omega^H}{\partial x}. \tag{5.10b}$$

6. Intrinsic plate functions for various plate problems

In general, the Kirchhoff-shear-deformable plate bending relationships can be specialized to any plate problem by first determining the corresponding intrinsic plate functions that comply with the plate geometry and basic boundary conditions. For the mathematical requirement, these intrinsic plate functions will further contain a total of six constants of integration that can be established using the rest of the plate boundary conditions to completely define the plate problem. Herein by using Lévy's proposed method of solutions, we shall look into three basic plate problems to study the effect of transverse shear deformation using the above-derived bending relationships. These three basic plate problems are:

(a) simply supported rectangular plates;
(b) rectangular plates with two opposite simply supported edges;
(c) clamped rectangular plates.

Particularly for plate problems of type (b), the corresponding intrinsic plate functions and the associated constants have appeared before in the works of Wang et al. (1999, 2001), Reddy et al. (2001) and Lee et al. (2002); the authors however wish to present these quantities in a generic form for a more general implementation. Else for plate problems of types (a) and (c), the analysis of thick plates using the current approach have hitherto not been presented.

6.1. Simply supported rectangular plates (SSSS)

For a rectangular plate that is simply supported along $y = \pm b/2$ and $x = \pm a/2$ (as shown in Fig. 1), the types of loading considered herein are distributed bending moments along two opposite plate edges. The results of such plate problem will provide the basis of solutions for clamped rectangular plates by ensuring a priori that the distributed moment applied to the simply supported edges of a transversely loaded plate will result in zero normal rotation.

6.1.1. Distributed moments along $y = \pm b/2$

In view of the plate loading and support conditions and assuming symmetrical bending of plates about the x-axis ($M_{yy}|_{y=b/2} = M_{yy}|_{y=-b/2}$), one can obtain the corresponding intrinsic plate functions using Lévy's proposed method of solutions as

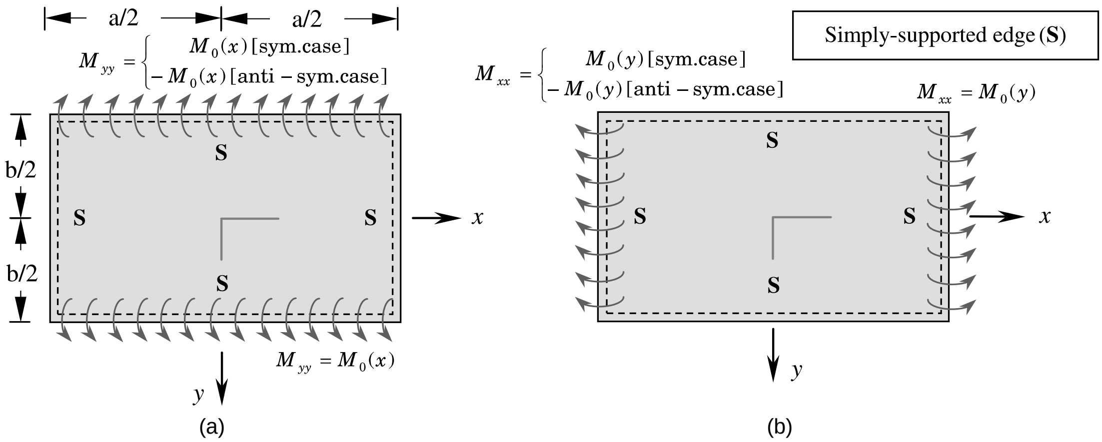

Fig. 1. Simply supported rectangular plate with edge distributed moments.

$$\begin{aligned} \Phi^H &= \sum_{m=1}^{\infty} \left(\frac{y}{2\alpha_m} C_{1m} \sinh \alpha_m y \right) \cos \alpha_m x, \\ \Psi^H &= \sum_{m=1}^{\infty} (C_{2m} \cosh \alpha_m y) \cos \alpha_m x, \\ \Omega^H &= \sum_{m=1}^{\infty} (C_{3m} \sinh \lambda_m y) \sin \alpha_m x, \end{aligned} \tag{6.1}$$

where $\lambda_m^2 = \alpha_m^2 + c^2$ and $\alpha_m = m\pi/a$. For the boundary conditions along $x = \pm a/2$, the index m is restricted to odd integers.

Now to determine the constants of integration, the boundary conditions along $y = \pm b/2$ are imposed as

$$w^H = w^K = 0, \quad \phi_x^H = \frac{\partial w^K}{\partial x} = 0, \quad M_{yy}^H = M_{yy}^K. \tag{6.2}$$

From Eqs. (5.3), (5.6a), (5.7b), (6.1) and (6.2), the constants can be solved as

$$\begin{aligned} C_{1m} &= 0, \\ C_{2m} &= -\frac{1}{K_s G h} \operatorname{sech} \frac{\alpha_m b}{2} \left(1 - \frac{\mathscr{B} c^2}{2} \right) \mathscr{M}_m^K|_{y=b/2}, \\ C_{3m} &= \frac{1}{K_s G h} \operatorname{sech} \frac{\lambda_m b}{2} \left(\frac{c^2 \alpha_m}{\mathscr{A} \lambda_m} \right) \mathscr{M}_m^K|_{y=b/2}. \end{aligned} \tag{6.3}$$

Now if the case of anti-symmetrical bending of plate about x-axis, i.e., $(M_{yy})_{y=b/2} = -(M_{yy})_{y=-b/2}$ is considered, the intrinsic plate functions will become

$$\begin{aligned} \Phi^H &= \sum_{m=1}^{\infty} \left(\frac{y}{2\alpha_m} C_{1m} \cosh \alpha_m y \right) \cos \alpha_m x, \\ \Psi^H &= \sum_{m=1}^{\infty} (C_{2m} \sinh \alpha_m y) \cos \alpha_m x, \\ \Omega^H &= \sum_{m=1}^{\infty} (C_{3m} \cosh \lambda_m y) \sin \alpha_m x. \end{aligned} \tag{6.4}$$

With the consideration of the boundary conditions as shown in Eq. (6.2), the constants are established as

$$\begin{aligned} C_{1m} &= 0, \\ C_{2m} &= -\frac{1}{K_s G h} \operatorname{cosech} \frac{\alpha_m b}{2} \left(1 - \frac{\mathscr{B} c^2}{2}\right) \mathscr{M}^K_m \big|_{y=b/2}, \\ C_{3m} &= \frac{1}{K_s G h} \operatorname{cosech} \frac{\lambda_m b}{2} \left(\frac{c^2 \alpha_m}{\mathscr{A} \lambda_m}\right) \mathscr{M}^K_m \big|_{y=b/2}, \end{aligned} \tag{6.5}$$

where $\mathscr{M}^K_m$ (defined as $\mathscr{M}^K = \sum \mathscr{M}^K_m \cos \alpha_m x$) can be computed from Eqs. (B.1) and (B.2d) or (B.3d). If one were to substitute the intrinsic plate functions with the evaluated constants into the deflection relationship Eq. (5.3) and together with the corresponding thin plate solution given in Eqs. (B.2) and (B.3), it can be shown for both cases of symmetrically and anti-symmetrically applied edge moments that

$$w^H = w^K. \tag{6.6}$$

Eq. (6.6) essentially applies to all three shear deformable plate theories considered; however, the rotations and stress resultants will differ from the corresponding thin plate solutions.

6.1.2. Distributed moments along $x = \pm a/2$

For distributed moments symmetrically applied along $x = \pm a/2$, the corresponding intrinsic plate functions are

$$\begin{aligned} \Phi^H &= \sum_{n=1}^{\infty} \left(\frac{x}{2\beta_n} D_{1n} \sinh \beta_n x\right) \cos \beta_n y, \\ \Psi^H &= \sum_{n=1}^{\infty} (D_{2n} \cosh \beta_n x) \cos \beta_n y, \\ \Omega^H &= \sum_{n=1}^{\infty} (D_{3n} \sinh \vartheta_n x) \sin \beta_n y, \end{aligned} \tag{6.7}$$

where $\vartheta_n^2 = \beta_n^2 + c^2$ and $\beta_n = n\pi/b$. Here, the index n is restricted to odd integers to comply with the boundary conditions along $y = \pm b/2$.

The boundary conditions at $x = \pm a/2$ for the problem will be

$$w^H = w^K = 0, \quad \phi_y^H = \frac{\partial w^K}{\partial y} = 0, \quad M_{xx}^H = M_{xx}^K. \tag{6.8}$$

Now by substituting Eqs. (6.7) and (6.8) into Eqs. (5.3), (5.6b) and (5.7a), one obtains

$$\begin{aligned} D_{1n} &= 0, \\ D_{2n} &= -\frac{1}{K_s G h} \operatorname{sech} \frac{\beta_n a}{2} \left(1 - \frac{\mathscr{B} c^2}{2}\right) \mathscr{M}^K_n \big|_{x=a/2}, \\ D_{3n} &= -\frac{1}{K_s G h} \operatorname{sech} \frac{\vartheta_n a}{2} \left(\frac{c^2 \beta_n}{\mathscr{A} \vartheta_n}\right) \mathscr{M}^K_n \big|_{x=a/2}. \end{aligned} \tag{6.9}$$

Similarly, for the case of anti-symmetrically applied edge moments along $x = \pm a/2$, the intrinsic plate functions are

$$\begin{aligned}
\Phi^H &= \sum_{n=1}^{\infty}\left(\frac{x}{2\beta_n}D_{1n}\cosh\beta_n x\right)\cos\beta_n y,\\
\Psi^H &= \sum_{n=1}^{\infty}(D_{2n}\sinh\beta_n x)\cos\beta_n y,\\
\Omega^H &= \sum_{n=1}^{\infty}(D_{3n}\cosh\vartheta_n x)\sin\beta_n y,
\end{aligned} \tag{6.10}$$

and the corresponding constants evaluated using the boundary conditions in Eq. (6.8) are

$$\begin{aligned}
D_{1n} &= 0,\\
D_{2n} &= -\frac{1}{K_s G h}\operatorname{cosech}\frac{\beta_n a}{2}\left(1-\frac{\mathscr{B}c^2}{2}\right)\mathscr{M}_n^K|_{x=a/2},\\
D_{3n} &= -\frac{1}{K_s G h}\operatorname{cosech}\frac{\vartheta_n a}{2}\left(\frac{c^2\beta_n}{\mathscr{A}\vartheta_n}\right)\mathscr{M}_n^K|_{x=a/2},
\end{aligned} \tag{6.11}$$

where $\mathscr{M}_n^K$ (given as $\mathscr{M}^K = \sum \mathscr{M}_n^K \sin\beta_n y$) can be calculated using Eqs. (B.4) and (B.5d) or (B.6d). As expected in both of these plate problems, the thick plate deflections for the three shear deformable plate theories will also be the same as the thin plate results.

6.2. Rectangular plates with two opposite simply supported edges

Consider a rectangular plate of length a and width b, that is simply supported along the two opposite edges ($x = 0, a$), as shown in Fig. 2. In view of the plate problem and adopting Lévy's proposed solutions, the intrinsic plate functions can be established as

$$\begin{aligned}
\Phi^H &= \sum_{m=1}^{\infty}\left[\frac{y}{2\alpha_m}(C_{1m}\cosh\alpha_m y + C_{2m}\sinh\alpha_m y)\right]\sin\alpha_m x,\\
\Psi^H &= \sum_{m=1}^{\infty}(C_{3m}\cosh\alpha_m y + C_{4m}\sinh\alpha_m y)\sin\alpha_m x,\\
\Omega^H &= \sum_{m=1}^{\infty}(C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\cos\alpha_m x,
\end{aligned} \tag{6.12}$$

where $\lambda_m^2 = \alpha_m^2 + c^2$ and $\alpha_m = m\pi/a$.

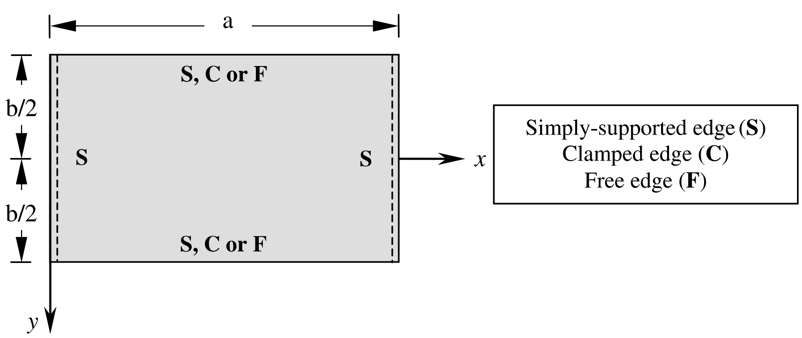

Fig. 2. Rectangular plate with two opposite simply supported edges.

6.2.1. Clamped edges along $y = \pm b/2$ (SCSC)

Now consider the case when the other two edges of the considered rectangular plate are clamped, one will impose the following boundary conditions along the edges ($y = \pm b/2$):

$$w^H = w^K = 0, \quad \phi_y^H = \frac{\partial w^K}{\partial y} = 0, \quad \phi_x^H = 0. \tag{6.13}$$

Hence by substituting Eqs. (6.12) and (6.13) into Eqs. (5.3), (5.6a) and (5.6b), the constants can be determined as

$$\begin{aligned}
C_{1m} &= \frac{\alpha_m\left[\left(\lambda_m \coth\frac{\alpha_m b}{2} - \alpha_m \coth\frac{\lambda_m b}{2}\right)\rho_m^- - \xi_m^- \coth\frac{\lambda_m b}{2}\right] + \lambda_m \eta_m^+}{\left\{\left[\frac{b}{4}\coth\frac{\alpha_m b}{2} - \alpha_m\left(\frac{\mathscr{D}}{\mathscr{A}}\right) - \frac{1}{2\alpha_m}\right]\lambda_m \cosh\frac{\alpha_m b}{2} - \left[\frac{\lambda_m b}{4} - \alpha_m^2\left(\frac{\mathscr{D}}{\mathscr{A}}\right)\coth\frac{\lambda_m b}{2}\right]\sinh\frac{\alpha_m b}{2}\right\}},\\
C_{2m} &= \frac{\alpha_m\left[\left(\lambda_m \tanh\frac{\alpha_m b}{2} - \alpha_m \tanh\frac{\lambda_m b}{2}\right)\rho_m^+ - \xi_m^+ \tanh\frac{\lambda_m b}{2}\right] + \lambda_m \eta_m^-}{\left\{\left[\frac{b}{4}\tanh\frac{\alpha_m b}{2} - \alpha_m\left(\frac{\mathscr{D}}{\mathscr{A}}\right) - \frac{1}{2\alpha_m}\right]\lambda_m \sinh\frac{\alpha_m b}{2} - \left[\frac{\lambda_m b}{4} - \alpha_m^2\left(\frac{\mathscr{D}}{\mathscr{A}}\right)\tanh\frac{\lambda_m b}{2}\right]\cosh\frac{\alpha_m b}{2}\right\}},\\
C_{3m} &= \operatorname{sech}\frac{\alpha_m b}{2}\left(\frac{b}{4\alpha_m}C_{2m}\sinh\frac{\alpha_m b}{2} - \rho_m^+\right),\\
C_{4m} &= \operatorname{cosech}\frac{\alpha_m b}{2}\left(\frac{b}{4\alpha_m}C_{1m}\cosh\frac{\alpha_m b}{2} - \rho_m^-\right),\\
C_{5m} &= -\left(\frac{c^2}{\lambda_m}\right)\operatorname{sech}\frac{\lambda_m b}{2}\left[\alpha_m\left(\frac{\mathscr{D}}{\mathscr{A}}\right)C_{2m}\cosh\frac{\alpha_m b}{2} + \alpha_m\rho_m^+ + \xi_m^+\right],\\
C_{6m} &= -\left(\frac{c^2}{\lambda_m}\right)\operatorname{cosech}\frac{\lambda_m b}{2}\left[\alpha_m\left(\frac{\mathscr{D}}{\mathscr{A}}\right)C_{1m}\sinh\frac{\alpha_m b}{2} + \alpha_m\rho_m^- + \xi_m^-\right],
\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}
\rho_m^{\pm} &= \frac{1}{2K_sGh}\left(1 - \frac{\mathscr{B}c^2}{2}\right)\left(\mathscr{M}_m^K|_{y=b/2} \pm \mathscr{M}_m^K|_{y=-b/2}\right),\\
\xi_m^{\pm} &= -\frac{1}{2K_sGh}\left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\left(Q_{xm}^K|_{y=b/2} \pm Q_{xm}^K|_{y=-b/2}\right),\\
\eta_m^{\pm} &= -\frac{1}{2K_sGh}\left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\left(Q_{ym}^K|_{y=b/2} \pm Q_{ym}^K|_{y=-b/2}\right),
\end{aligned} \tag{6.15}$$

and $\mathscr{M}_m^K$, Q_{xm}^K and Q_{ym}^K can be determined using Eqs. (B.7), (B.8) and (B.9b).

6.2.2. Free edges along $y = \pm b/2$ (SFSF)

For the case of free edges along $y = \pm b/2$, the boundary conditions to be defined are

$$M_{yy}^H = M_{yy}^K = 0, \quad Q_y^H = V_y^K = 0, \quad M_{xy}^H = 0. \tag{6.16}$$

In view of Eqs. (6.12), (6.16), (5.7b), (5.8) and (5.9b), one obtains the constants as

$$
\begin{aligned}
C_{1m} &= \frac{\delta_m^- + \left[\lambda_m \tanh\frac{\lambda_m b}{2} - \frac{1}{2\alpha_m}(\lambda_m^2+\alpha_m^2)\tanh\frac{\alpha_m b}{2}\right]\rho_m^+ + (\xi_m^+ + \eta_m^+)\tanh\frac{\alpha_m b}{2}}{\left\{\left(\frac{b}{4} - \lambda_m \mathscr{D}\tanh\frac{\lambda_m b}{2}\right)\alpha_m\cosh\frac{\alpha_m b}{2} - \left[\frac{\alpha_m b}{4}\tanh\frac{\alpha_m b}{2} - \alpha_m^2\mathscr{D} - \frac{3+v}{2(1-v)}\right]\sinh\frac{\alpha_m b}{2}\right\}},\\
C_{2m} &= \frac{\delta_m^+ + \left[\lambda_m \coth\frac{\lambda_m b}{2} - \frac{1}{2\alpha_m}(\lambda_m^2+\alpha_m^2)\coth\frac{\alpha_m b}{2}\right]\rho_m^- + (\xi_m^- + \eta_m^-)\coth\frac{\alpha_m b}{2}}{\left\{\left(\frac{b}{4} - \lambda_m \mathscr{D}\coth\frac{\lambda_m b}{2}\right)\alpha_m\sinh\frac{\alpha_m b}{2} - \left[\frac{\alpha_m b}{4}\coth\frac{\alpha_m b}{2} - \alpha_m^2\mathscr{D} - \frac{3+v}{2(1-v)}\right]\cosh\frac{\alpha_m b}{2}\right\}},\\
C_{3m} &= \frac{1}{\alpha_m^2}\operatorname{cosech}\frac{\alpha_m b}{2}\left\{\left[\frac{\alpha_m b}{4}\cosh\frac{\alpha_m b}{2} - \frac{1+v}{2(1-v)}\sinh\frac{\alpha_m b}{2}\right]C_{2m} - \frac{1}{2\alpha_m}(\lambda_m^2+\alpha_m^2)\rho_m^- + \xi_m^- + \eta_m^-\right\},\\
C_{4m} &= \frac{1}{\alpha_m^2}\operatorname{sech}\frac{\alpha_m b}{2}\left\{\left[\frac{\alpha_m b}{4}\sinh\frac{\alpha_m b}{2} - \frac{1+v}{2(1-v)}\cosh\frac{\alpha_m b}{2}\right]C_{1m} - \frac{1}{2\alpha_m}(\lambda_m^2+\alpha_m^2)\rho_m^+ + \xi_m^+ + \eta_m^+\right\},\\
C_{5m} &= -\frac{c^2}{\mathscr{A}}\operatorname{cosech}\frac{\lambda_m b}{2}\left(\mathscr{D}C_{2m}\sinh\frac{\alpha_m b}{2} + \frac{\rho_m^-}{\alpha_m}\right),\\
C_{6m} &= -\frac{c^2}{\mathscr{A}}\operatorname{sech}\frac{\lambda_m b}{2}\left(\mathscr{D}C_{1m}\cosh\frac{\alpha_m b}{2} + \frac{\rho_m^+}{\alpha_m}\right),
\end{aligned}
\tag{6.17}
$$

where

$$
\begin{aligned}
\rho_m^\pm &= \frac{1}{2K_s Gh}(Q_{ym}^K|_{y=b/2} \pm Q_{ym}^K|_{y=-b/2}),\\
\xi_m^\pm &= \frac{1}{2D(1-v)}(M_{xym}^K|_{y=b/2} \pm M_{xym}^K|_{y=-b/2}),\\
\eta_m^\pm &= \frac{\alpha_m}{2K_s Gh}\left(\frac{\mathscr{B}c^2}{2}\right)(Q_{ym}^K|_{y=b/2} \pm Q_{ym}^K|_{y=-b/2}),\\
\delta_m^\pm &= -\frac{\alpha_m}{2K_s Gh}\left(\frac{\mathscr{B}c^2}{2}\right)(Q_{xm}^K|_{y=b/2} \pm Q_{xm}^K|_{y=-b/2}),
\end{aligned}
\tag{6.18}
$$

and Eqs. (B.7), (B.8) and (B.9c) shall be used to establish M_{xym}^K, Q_{xm}^K and Q_{ym}^K.

6.3. Clamped rectangular plates (CCCC)

Treating the clamped plate problem, as shown in Fig. 3, it is assumed that the bending of the plate is to be symmetrical about both axes. To obtain the solutions for clamped plates, the approach will be to superimpose the solutions of simply supported plates under transverse loads and the corresponding solutions for simply supported plates with distributed moments along the plate edges.

In view of the symmetrical bending of plate, the intrinsic plate functions are

$$
\begin{aligned}
\Phi^H &= \sum_{m=1}^{\infty}\left(\frac{y}{2\alpha_m}C_{1m}\sinh\alpha_m y\right)\cos\alpha_m x + \sum_{n=1}^{\infty}\left(\frac{x}{2\beta_n}D_{1n}\sinh\beta_n x\right)\cos\beta_n y,\\
\Psi^H &= \sum_{m=1}^{\infty}(C_{2m}\cosh\alpha_m y)\cos\alpha_m x + \sum_{n=1}^{\infty}(D_{2n}\cosh\beta_n x)\cos\beta_n y,\\
\Omega^H &= \sum_{m=1}^{\infty}(C_{3m}\sinh\lambda_m y)\sin\alpha_m x + \sum_{n=1}^{\infty}(D_{3n}\sinh\vartheta_n x)\sin\beta_n y,
\end{aligned}
\tag{6.19}
$$

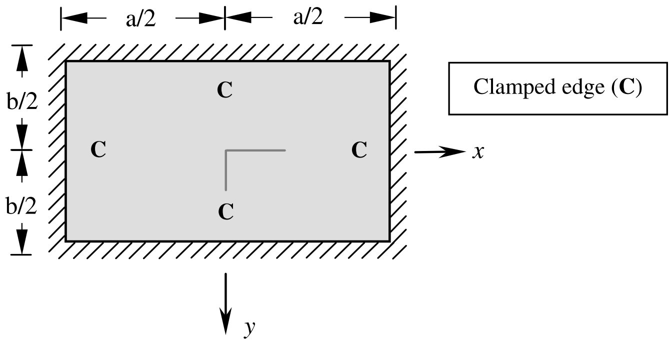

Fig. 3. Clamped rectangular plate.

where $\lambda_m^2 = \alpha_m^2 + c^2$ and $\alpha_m = m\pi/a$ while $\vartheta_m^2 = \beta_n^2 + c^2$ and $\beta_n = n\pi/b$ and the indices m and n only take odd integers. It is to note that the intrinsic plate functions for the present plate problem can be established by simply summing up the corresponding functions of the transversely loaded simply supported plates which are identically zero (Wang et al., 1999) and those of simply supported plates with distributed edge moments, Eqs. (6.1) and (6.7).

With Eqs. (5.3), (5.6a), (5.6b) and (6.19), one can write the specialized deflection and rotation–slope relationships of a clamped plate as

$$w^H = w^K + \left(1 - \frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}^K}{K_s G h} + \sum_{m=1}^{\infty}\left(C_{2m}\cosh\alpha_m y - \frac{y}{2\alpha_m}C_{1m}\sinh\alpha_m y\right)\cos\alpha_m x + \sum_{n=1}^{\infty}\left(D_{2n}\cosh\beta_n x - \frac{x}{2\beta_n}D_{1n}\sinh\beta_n x\right)\cos\beta_n y, \tag{6.20}$$

$$\begin{aligned}\phi_x^H = {} & -\frac{\partial w^K}{\partial x} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_x^K}{K_s G h} - \sum_{m=1}^{\infty}\left\{\alpha_m\left[\frac{y}{2\alpha_m}C_{1m}\sinh\alpha_m y + \left(\frac{\mathscr{D}}{\mathscr{A}}C_{1m} - C_{2m}\right)\cosh\alpha_m y\right]\right.\\ & \left. - \frac{\lambda_m}{c^2}C_{3m}\cosh\lambda_m y\right\}\sin\alpha_m x + \sum_{n=1}^{\infty}\left\{\beta_n\left[\left(\frac{\mathscr{D}}{\mathscr{A}}D_{1n} - D_{2n} + \frac{1}{2\beta_n^2}D_{1n}\right)\sinh\beta_n x + \frac{x}{2\beta_n}D_{1n}\cosh\beta_n x\right.\right.\\ & \left.\left. + \frac{1}{c^2}D_{3n}\sinh\vartheta_n x\right]\right\}\cos\beta_n y,\end{aligned} \tag{6.21a}$$

$$\begin{aligned}\phi_y^H = {} & -\frac{\partial w^K}{\partial y} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_y^K}{K_s G h} - \sum_{n=1}^{\infty}\left\{\beta_n\left[\frac{x}{2\beta_n}D_{1n}\sinh\beta_n x + \left(\frac{\mathscr{D}}{\mathscr{A}}D_{1n} - D_{2n}\right)\cosh\beta_n x\right]\right.\\ & \left. + \frac{\vartheta_n}{c^2}D_{3n}\cosh\vartheta_n x\right\}\sin\beta_n y + \sum_{m=1}^{\infty}\left\{\alpha_m\left[\left(\frac{\mathscr{D}}{\mathscr{A}}C_{1m} - C_{2m} + \frac{1}{2\alpha_m^2}C_{1m}\right)\sinh\alpha_m y\right.\right.\\ & \left.\left. + \frac{y}{2\alpha_m}C_{1m}\cosh\alpha_m y - \frac{1}{c^2}C_{3m}\sinh\lambda_m y\right]\right\}\cos\alpha_m x.\end{aligned} \tag{6.21b}$$

To satisfy the boundary edge conditions of a clamped plate using Eqs. (3.10), (4.6), (6.20), (6.21a) and (6.21b), the constants, C_{im} and D_{in} $(i = 1, 2, 3)$ are determined as

$$
\begin{aligned}
&-\frac{4\beta_n}{b}\sin\frac{n\pi}{2}\sum_{m=1}^{\infty}\left\{\alpha_m\left[\eta_{mn}C_{1m}\cosh\frac{\alpha_m b}{2}+\delta_{mn}c^2\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_m^K|_{y=b/2}}{K_sGh}\right]\right.\\
&\quad\left.-\frac{1}{\alpha_m^2+\vartheta_n^2}\left(1-\frac{1}{\mathscr{A}}-\frac{\mathscr{B}c^2}{2}\right)\frac{Q_{xm}^K|_{y=b/2}}{K_sGh}\right\}\sin\frac{m\pi}{2}+\left[\beta_n\frac{\mathscr{D}}{\mathscr{A}}\left(\tanh\frac{\beta_n a}{2}-\frac{\beta_n}{\vartheta_n}\tanh\frac{\vartheta_n a}{2}\right)\right.\\
&\quad\left.+\frac{1}{2\beta_n}\tanh\frac{\beta_n a}{2}+\frac{a}{4}\operatorname{sech}^2\frac{\beta_n a}{2}\right]D_{1n}\cosh\frac{\beta_n a}{2}\\
&=\left(1-\frac{1}{\mathscr{A}}-\frac{\mathscr{B}c^2}{2}\right)\left(\frac{Q_{xn}^K|_{x=a/2}}{K_sGh}+\frac{\beta_n}{\vartheta_n}\tanh\frac{\vartheta_n a}{2}\frac{Q_{yn}^K|_{x=a/2}}{K_sGh}\right)-\beta_n\left(\tanh\frac{\beta_n a}{2}-\frac{\beta_n}{\vartheta_n}\tanh\frac{\vartheta_n a}{2}\right)\\
&\quad\times\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_n^K|_{x=a/2}}{K_sGh},
\end{aligned}
\tag{6.22a}
$$

$$
\begin{aligned}
&-\frac{4\alpha_m}{a}\sin\frac{m\pi}{2}\sum_{n=1}^{\infty}\left\{\beta_n\left[\eta_{mn}D_{1n}\cosh\frac{\beta_n a}{2}+\delta_{mn}c^2\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_n^K|_{x=a/2}}{K_sGh}\right]\right.\\
&\quad\left.-\frac{1}{\alpha_m^2+\vartheta_n^2}\left(1-\frac{1}{\mathscr{A}}-\frac{\mathscr{B}c^2}{2}\right)\frac{Q_{yn}^K|_{x=a/2}}{K_sGh}\right\}\sin\frac{n\pi}{2}+\left[\alpha_m\frac{\mathscr{D}}{\mathscr{A}}\left(\tanh\frac{\alpha_m b}{2}-\frac{\alpha_m}{\lambda_m}\tanh\frac{\lambda_m b}{2}\right)\right.\\
&\quad\left.+\frac{1}{2\alpha_m}\tanh\frac{\alpha_m b}{2}+\frac{b}{4}\operatorname{sech}^2\frac{\alpha_m b}{2}\right]C_{1m}\cosh\frac{\alpha_m b}{2}\\
&=\left(1-\frac{1}{\mathscr{A}}-\frac{\mathscr{B}c^2}{2}\right)\left(\frac{Q_{ym}^K|_{y=b/2}}{K_sGh}+\frac{\alpha_m}{\lambda_m}\tanh\frac{\lambda_m b}{2}\frac{Q_{xm}^K|_{y=b/2}}{K_sGh}\right)\\
&\quad-\alpha_m\left(\tanh\frac{\alpha_m b}{2}-\frac{\alpha_m}{\lambda_m}\tanh\frac{\lambda_m b}{2}\right)\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_m^K|_{y=b/2}}{K_sGh},
\end{aligned}
\tag{6.22b}
$$

$$
C_{2m}=\operatorname{sech}\frac{\alpha_m b}{2}\left[C_{1m}\frac{b}{4\alpha_m}\sinh\frac{\alpha_m b}{2}-\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_m^K|_{y=b/2}}{K_sGh}\right],
\tag{6.22c}
$$

$$
C_{3m}=\frac{c^2}{\lambda_m}\operatorname{sech}\frac{\lambda_m b}{2}\left\{\alpha_m\left[\frac{\mathscr{D}}{\mathscr{A}}C_{1m}\cosh\frac{\alpha_m b}{2}+\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_m^K|_{y=b/2}}{K_sGh}\right]+\left(1-\frac{1}{\mathscr{A}}-\frac{\mathscr{B}c^2}{2}\right)\frac{Q_{xm}^K|_{y=b/2}}{K_sGh}\right\},
\tag{6.22d}
$$

$$
D_{2n}=\operatorname{sech}\frac{\beta_n a}{2}\left[D_{1n}\frac{a}{4\beta_n}\sinh\frac{\beta_n a}{2}-\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_n^K|_{x=a/2}}{K_sGh}\right],
\tag{6.22e}
$$

$$
D_{3n}=-\frac{c^2}{\vartheta_n}\operatorname{sech}\frac{\vartheta_n a}{2}\left\{\beta_n\left[\frac{\mathscr{D}}{\mathscr{A}}D_{1n}\cosh\frac{\beta_n a}{2}+\left(1-\frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}_n^K|_{x=a/2}}{K_sGh}\right]+\left(1-\frac{1}{\mathscr{A}}-\frac{\mathscr{B}c^2}{2}\right)\frac{Q_{yn}^K|_{x=a/2}}{K_sGh}\right\},
\tag{6.22f}
$$

where $\mathscr{M}_m^K$, Q_{xm}^K and Q_{ym}^K are evaluated from Eqs. (B.1) and (B.2) and $\mathscr{M}_n^K$, Q_{xn}^K and Q_{yn}^K are evaluated from Eqs. (B.4) and (B.5) while

$$\eta_{mn} = -\frac{1}{(\alpha_m^2 + \beta_n^2)^2} + \frac{2\delta_{mn}}{\mathscr{A}(1-\nu)} \quad \text{and} \quad \delta_{mn} = \frac{1}{(\alpha_m^2 + \beta_n^2)(\alpha_m^2 + \vartheta_n^2)}.$$

It should be noted that the expressions in Eqs. (6.22a–f) are not in an *explicit* form in the sense that the evaluation of C_{1m} and D_{1n} is dependent on one another. This will pose a difficulty as such a dependency of the constants requires one to solve an infinite system of equations. A commonly adopted approach is to reduce it to a finite system of equations by considering only a finite number of terms. Such an approximation of the system will then raise several mathematical issues such as the convergence of solutions and their uniqueness. These issues were addressed by Meleshko (1997) in his findings where he reported that for thin clamped plate solutions, the resulting infinite system of equations is *regular* and will therefore have a set of unique and bounded solutions. Furthermore, the solution of the reduced finite system using the method of successive reduction will be convergent, tending towards the unique solution of the infinite system. While the proof is not provided in this study, numerical studies will nonetheless be carried out to ensure the convergence and accuracy of the solutions.

7. Results and discussion

With the intrinsic plate functions and their corresponding constants determined for the respective plate problems, one can now utilize the bending relationships as derived earlier to generate thick plate results. As discussed earlier, the convergence of the series solution will be of concern for the case of clamped plates. Table 1 shows the numerical results from a convergence study to determine the appropriate number of terms needed for thick plate solutions with an acceptable degree of precision using the three different shear deformable plate theories. The plate problem concerned herein is a clamped square plate under uniformly distributed load q_0. It is clear that the results for the maximum deflection show faster convergence than those for the bending moments and a good precision can be noted in the solutions when thirty or more terms have been adopted for the series. With that from here on, the number of terms that will be used for computation is 20, unless specified otherwise.

To verify the accuracy or correctness of the relationships, available analytical results found in the open literature and those generated by ABAQUS will be used for comparison. Table 2 presents the maximum transverse deflections of square Mindlin plates of various boundary conditions and plate thicknesses subjected to uniformly distributed load q_0 computed by the bending relationships and ABAQUS. It is to note that a mesh size of 40 by 40 thick plate (S8R) elements has been used for generating ABAQUS solutions. One can see from the table that there is an excellent agreement between the present results and those furnished by ABAQUS.

Now adopting the Reissner plate theory, transverse deflections and the shear forces computed by the relationships for SFSF thick plates have been tabulated in Table 3, together with the solution as given by Salerno and Goldberg (1960). A one-to-one correspondence between the two sets of results is observed.

Levinson plate results generated via the bending relationships had already been presented in the previous work of the authors (Reddy et al., 2001) and compared with those computed by Levinson and Cooke (1983) and Cooke and Levinson (1983) for all-round simply supported plates and plates with two opposite simply supported edges. In that study, the solutions for simply supported plates had matched up very well while the results for the second plate problem however showed notable dissimilarity. It was shown then that there are missing terms in their governing equations, leading to erroneous numerical results thereafter. For a more detailed discussion of the topic, readers may refer to the above cited references.

To review the differences of the predictions for the thick plate results of the three shear deformable plate theories, maximum normalized transverse deflections of clamped rectangular plates under a uniformly distributed load have been furnished in Table 4. From Table 4, one can see that of the three, the Levinson

Table 1
Convergence studies for the thick plate results of clamped square plate under uniformly distributed load q_0 for the various shear deformable plate theories ($h/a = 0.1$, $K_s = 5/6$, $\nu = 0.3$)

No. of terms (m, n)	$(\bar{w}_{max})^a$			$(\overline{M}_{xx})^a$		
	Mindlin	Reissner	Levinson	Mindlin	Reissner	Levinson
5	1.5057	1.5046	1.5626	5.0582	5.0462	5.0268
10	1.5047	1.5042	1.5624	5.0740	5.0620	5.0402
20	1.5046	1.5044	1.5628	5.0738	5.0614	5.0397
30	1.5046	1.5045	1.5630	5.0740	5.0615	5.0398
40	1.5046	1.5045	1.5630	5.0742	5.0616	5.0399
50	1.5046	1.5045	1.5631	5.0743	5.0616	5.0399

[a] $\bar{w}_{max} = 1000w(0,0)D/(q_0a^4)$, $\overline{M}_{xx} = -100M_{xx}(a/2,0)/(q_0a^2)$.

Table 2
Normalized maximum plate deflections $[1000wD/(q_0a^4)]$ of square Mindlin plates with various boundary conditions and plate thicknesses under uniformly distributed load q_0 ($\mathscr{A} = 1$, $\mathscr{B} = 0$; $K_s = 5/6$, $\nu = 0.3$)

h/a	SCSC		SFSF		CCCC	
	ABAQUS	Present results	ABAQUS	Present results	ABAQUS	Present results
0.005	1.9179	**1.9179**	15.0237	**15.0237**	1.2660	**1.2660**
0.01	1.9202	**1.9202**	15.0380	**15.0380**	1.2679	**1.2679**
0.05	1.9918	**1.9918**	15.2165	**15.2165**	1.3273	**1.3273**
0.1	2.2087	**2.2087**	15.6001	**15.6001**	1.5046	**1.5046**
0.2	3.0211	**3.0211**	16.8975	**16.8975**	2.1722	**2.1722**

Table 3
Normalized plate results of square Reissner plates (SFSF) with different plate thicknesses under uniformly distributed load q_0 ($\mathscr{A} = 1$, $\mathscr{B} = h^2\nu/10$; $K_s = 5/6$, $\nu = 0.3$)

h/a	$\bar{w}(a/2, a/2)$		$\overline{Q}_x(0,0)$		$\overline{Q}_y(a/2, a/4)$	
	Salerno and Goldberg (1960)	Present results	Salerno and Goldberg (1960)	Present results	Salerno and Goldberg (1960)	Present results
0.005	15.0236	**15.0236**	4.6343	**4.6343**	2.6503	**2.6503**
0.01	15.0376	**15.0376**	4.6325	**4.6325**	2.6651	**2.6651**
0.05	15.2081	**15.2081**	4.6167	**4.6167**	2.7875	**2.7875**
0.1	15.5677	**15.5677**	4.5954	**4.5954**	2.9479	**2.9479**
0.2	16.7760	**16.7760**	4.5499	**4.5499**	3.1617	**3.1617**

$\bar{w} = 1000wD/(q_0a^4)$, $\overline{Q}_x = 10Q_x/(q_0a)$, $\overline{Q}_y = -100Q_y/(q_0a)$.

plate theory consistently predicts the highest deflections for all cases; the most notable differences are those for the very thick plates. This is to be expected since in the Levinson plate theory, the formulation allows the normals to the mid-plane to warp through the plate thickness rendering the plate to be relatively *flexible*. However, the more interesting observation is the stiffer solution yielded by the Reissner plate theory which allows the consideration of the normal stress (σ_{zz}) and the plate normal to deform; this marks the distinct differences between the Reissner and Mindlin plate theories and one can refer to Wang et al. (2001) and Lee et al. (2002) for more detailed discussion. The stiffer behavior of the Reissner plate results

Table 4
Maximum normalized plate deflections $[1000wD/(q_0a^4)]$ of a clamped rectangular plate with various plate thickness under uniformly distributed load q_0, specialized for the various shear deformable plate theories ($K_s = 5/6$, $\nu = 0.3$)

h/a	$b/a = 1.0$			$b/a = 1.5$			$b/a = 2.0$		
	MPT	RPT	LPT	MPT	RPT	LPT	MPT	RPT	LPT
0.005	1.2660	1.2660	1.2661	2.1974	2.1974	2.1976	2.5339	2.5339	2.5341
0.01	1.2679	1.2679	1.2685	2.1999	2.1999	2.2008	2.5366	2.5366	2.5375
0.05	1.3273	1.3272	1.3425	2.2802	2.2801	2.3008	2.6236	2.6235	2.6460
0.1	1.5046	1.5044	1.5628	2.5246	2.5241	2.6050	2.8927	2.8921	2.9814
0.2	2.1722	2.1711	2.3911	3.4610	3.4591	3.7675	3.9473	3.9452	4.2941

MPT = Mindlin plate theory, RPT = Reissner plate theory and LPT = Levinson plate theory.

Table 5
Normalized stress resultants of clamped square plates with various plate thickness under uniformly distributed load q_0, specialized for the various shear deformable plate theories ($K_s = 5/6$, $\nu = 0.3$)

h/a	$(\overline{M}_{xx})^a$			$(\overline{M}_{xy})^a$			$(\overline{Q}_y)^a$		
	MPT	RPT	LPT	MPT	RPT	LPT	MPT	RPT	LPT
0.005	5.1332	5.1332	5.1331	0.0007	0.0007	0.0011	44.0202	44.0205	44.0204
0.01	5.1327	5.1326	5.1323	0.0013	0.0014	0.0024	43.9145	43.9150	43.9125
0.05	5.1166	5.1135	5.1072	−0.0507	−0.0456	−0.0469	42.8595	42.8606	42.7887
0.1	5.0738	5.0614	5.0397	−0.1403	−0.1185	−0.1690	41.1982	41.1969	40.9739
0.2	4.9797	4.9299	4.8719	−0.2647	−0.1755	−0.5530	38.1903	38.1796	37.8461

Classical thin plate solution: $\overline{M}^K_{xx} = 5.1334$, $\overline{M}^K_{xy} = -0.0018$, $\overline{Q}^K_y = 44.1193$.
[a] $\overline{M}_{xx} = -100M_{xx}(a/2,0)/(q_0a^2)$, $\overline{M}_{xy} = 100M_{xy}(a/2,b/2)/(q_0a^2)$, $\overline{Q}_y = -100Q_y(0,b/2)/(q_0a)$.

Table 6
Normalized stress resultants of clamped rectangular plates with various plate thickness under uniformly distributed load q_0, specialized for the various shear deformable plate theories ($b/a = 1.5$, $K_s = 5/6$, $\nu = 0.3$)

h/a	$(\overline{M}_{xx})^a$			$(\overline{M}_{xy})^a$			$(\overline{Q}_y)^a$		
	MPT	RPT	LPT	MPT	RPT	LPT	MPT	RPT	LPT
0.005	7.5661	7.5661	7.5661	0.0010	0.0011	0.0014	46.4041	46.4044	46.4040
0.01	7.5659	7.5658	7.5656	0.0019	0.0020	0.0030	46.2738	46.2744	46.2708
0.05	7.5586	7.5555	7.5514	−0.0532	−0.0475	−0.0491	44.9630	44.9640	44.8717
0.1	7.5343	7.5218	7.5049	−0.1588	−0.1341	−0.1833	42.9585	42.9568	42.6937
0.2	7.4470	7.3969	7.3366	−0.3191	−0.2179	−0.5818	39.5971	39.5854	39.2509

Classical thin plate solution: $\overline{M}^K_{xx} = 7.5662$, $\overline{M}^K_{xy} = -0.0016$, $\overline{Q}^K_y = 46.5255$.
[a] $\overline{M}_{xx} = -100M_{xx}(a/2,0)/(q_0a^2)$, $\overline{M}_{xy} = 100M_{xy}(a/2,b/2)/(q_0a^2)$, $\overline{Q}_y = -100Q_y(0,b/2)/(q_0a)$.

may be attributed to the weighted average approach that has been introduced to give *equivalent* values of mid-plane transverse displacement and normal rotations. Nonetheless, these dissimilarities in the formulation for the two plate theories and their results, although small for thin plates, should not be ignored and hence the two theories should not be treated as the same. To illustrate the variation of the stress resultants predicted by the three plate theories for clamped plates with various plate aspect ratios (b/a) and plate

Table 7
Normalized stress resultants of clamped rectangular plates with various plate thickness under uniformly distributed load q_0, specialized for the various shear deformable plate theories ($b/a = 2.0$, $K_s = 5/6$, $\nu = 0.3$)

h/a	$(\overline{M}_{xx})^a$			$(\overline{M}_{xy})^a$			$(\overline{Q}_y)^a$		
	MPT	RPT	LPT	MPT	RPT	LPT	MPT	RPT	LPT
0.005	8.2875	8.2875	8.2875	0.0023	0.0023	0.0027	46.2647	46.2650	46.2641
0.01	8.2875	8.2874	8.2873	0.0029	0.0031	0.0040	46.1295	46.1301	46.1258
0.05	8.2860	8.2828	8.2802	−0.0556	−0.0495	−0.0526	44.8010	44.8019	44.7072
0.1	8.2798	8.2672	8.2563	−0.1708	−0.1439	−0.1977	42.7661	42.7643	42.4935
0.2	8.2449	8.1946	8.1479	−0.3478	−0.2367	−0.6043	39.3523	39.3407	38.9945

Classical thin plate solution: $\overline{M}_{xx}^K = 8.2875$, $\overline{M}_{xy}^K = -0.0005$, $\overline{Q}_y^K = 46.3913$.

[a] $\overline{M}_{xx} = -100M_{xx}(a/2,0)/(q_0a^2)$, $\overline{M}_{xy} = 100M_{xy}(a/2,b/2)/(q_0a^2)$, $\overline{Q}_y = -100Q_y(0,b/2)/(q_0a)$.

thicknesses (h), computed solutions are presented in Tables 5–7. These should serve as references for numerical solutions.

8. Concluding remarks

Presented herein are the canonical bending relationships that allow one to generate thick plate results for the Mindlin, Reissner and Levinson plate theories using the widely available thin plate solutions. The bending relationships have been derived for several plate problems like simply supported rectangular plates subjected to distributed edge moments, rectangular plates with two simply supported edges and clamped rectangular plates both subjected to uniformly distributed transverse loads. A convergence study has been carried out to determine the necessary number of terms for attaining an acceptable precision for the thick plate results. Also, the correctness of the thick plate results furnished by the canonical bending relationships for the various plate problems had been substantiated by ABAQUS and other existing results.

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Appendix A. Specialized bending relationships

For the ease of the application of the Kirchhoff-shear-deformable bending relationships derived in the main text, these relationships have been specialized in this appendix for each of the plate problems being considered herein this study. Together with the evaluation of the corresponding constants and the thin plate solutions (as given in Appendix B), these relationships can be readily programmed to generate useful thick plate results.

A.1. Simply supported rectangular plates

A.1.1. Distributed moments along $y = \pm b/2$

(a) *Symmetrical case*. $M_{yy}|_{y=b/2} = M_{yy}|_{y=-b/2}$

$$\begin{aligned}
w^H &= w^K + \left(1 - \frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}^K}{K_sGh} + \sum_{m=1}^{\infty}(C_{2m}\cosh\alpha_m y)\cos\alpha_m x,\\
\phi_x^H &= -\frac{\partial w^K}{\partial x} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_x^K}{K_sGh} + \sum_{m=1}^{\infty}\alpha_m\left(C_{2m}\cosh\alpha_m y + \frac{\lambda_m}{c^2\alpha_m}C_{3m}\cosh\lambda_m y\right)\sin\alpha_m x,\\
\phi_y^H &= -\frac{\partial w^K}{\partial y} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_y^K}{K_sGh} - \sum_{m=1}^{\infty}\alpha_m\left(C_{2m}\sinh\alpha_m y + \frac{1}{c^2}C_{3m}\sinh\lambda_m y\right)\cos\alpha_m x,\\
M_{xx}^H &= M_{xx}^K - \mathscr{B}\frac{\partial Q_y^k}{\partial y} + D(1-\nu)\sum_{m=1}^{\infty}\alpha_m^2\left(C_{2m}\cosh\alpha_m y + \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}C_{3m}\cosh\lambda_m y\right)\cos\alpha_m x,\\
M_{yy}^H &= M_{yy}^K - \mathscr{B}\frac{\partial Q_x^k}{\partial x} - D(1-\nu)\sum_{m=1}^{\infty}\alpha_m^2\left(C_{2m}\cosh\alpha_m y + \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}C_{3m}\cosh\lambda_m y\right)\cos\alpha_m x,\\
M_{xy}^H &= M_{xy}^K + \mathscr{B}\frac{\partial Q_y^k}{\partial x} + D(1-\nu)\sum_{m=1}^{\infty}\left[\alpha_m^2 C_{2m}\sinh\alpha_m y + \frac{\mathscr{A}}{2c^2\alpha_m}(\lambda_m^2+\alpha_m^2)C_{3m}\sinh\lambda_m y\right]\sin\alpha_m x,\\
Q_x^H &= Q_x^K + \frac{\mathscr{A}}{2}D(1-\nu)\sum_{m=1}^{\infty}(\lambda_m C_{3m}\cosh\lambda_m y)\sin\alpha_m x,\\
Q_y^H &= Q_y^K - \frac{\mathscr{A}}{2}D(1-\nu)\sum_{m=1}^{\infty}(\alpha_m C_{3m}\sinh\lambda_m y)\cos\alpha_m x.
\end{aligned} \tag{A.1}$$

(b) *Anti-symmetrical case*. $M_{yy}|_{y=b/2} = -M_{yy}|_{y=-b/2}$

$$\begin{aligned}
w^H &= w^K + \left(1 - \frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}^K}{K_sGh} + \sum_{m=1}^{\infty}(C_{2m}\sinh\alpha_m y)\cos\alpha_m x,\\
\phi_x^H &= -\frac{\partial w^K}{\partial x} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_x^K}{K_sGh} + \sum_{m=1}^{\infty}\alpha_m\left(C_{2m}\sinh\alpha_m y + \frac{\lambda_m}{c^2\alpha_m}C_{3m}\sinh\lambda_m y\right)\sin\alpha_m x,\\
\phi_y^H &= -\frac{\partial w^K}{\partial y} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_y^K}{K_sGh} - \sum_{m=1}^{\infty}\alpha_m\left(C_{2m}\cosh\alpha_m y + \frac{1}{c^2}C_{3m}\cosh\lambda_m y\right)\cos\alpha_m x,\\
M_{xx}^H &= M_{xx}^K - \mathscr{B}\frac{\partial Q_y^K}{\partial y} + D(1-\nu)\sum_{m=1}^{\infty}\alpha_m^2\left(C_{2m}\sinh\alpha_m y + \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}C_{3m}\sinh\lambda_m y\right)\cos\alpha_m x,\\
M_{yy}^H &= M_{yy}^K - \mathscr{B}\frac{\partial Q_x^K}{\partial x} - D(1-\nu)\sum_{m=1}^{\infty}\alpha_m^2\left(C_{2m}\sinh\alpha_m y + \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}C_{3m}\sinh\lambda_m y\right)\cos\alpha_m x,\\
M_{xy}^H &= M_{xy}^K + \mathscr{B}\frac{\partial Q_y^K}{\partial x} + D(1-\nu)\sum_{m=1}^{\infty}\left[\alpha_m^2 C_{2m}\cosh\alpha_m y + \frac{\mathscr{A}}{2c^2\alpha_m}(\lambda_m^2+\alpha_m^2)C_{3m}\cosh\lambda_m y\right]\sin\alpha_m x,\\
Q_x^H &= Q_x^K + \frac{\mathscr{A}}{2}D(1-\nu)\sum_{m=1}^{\infty}(\lambda_m C_{3m}\sinh\lambda_m y)\sin\alpha_m x,\\
Q_y^H &= Q_y^K - \frac{\mathscr{A}}{2}D(1-\nu)\sum_{m=1}^{\infty}(\alpha_m C_{3m}\cosh\lambda_m y)\cos\alpha_m x.
\end{aligned} \tag{A.2}$$

A.1.2. Distributed moments along $x = \pm a/2$

(a) *Symmetrical case.* $M_{xx}|_{x=a/2} = M_{xx}|_{x=-a/2}$

$$
\begin{aligned}
w^H &= w^K + \left(1 - \frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}^K}{K_s Gh} + \sum_{n=1}^{\infty}(D_{2n}\cosh\beta_n x)\cos\beta_n y,\\
\phi_x^H &= -\frac{\partial w^K}{\partial x} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_x^K}{K_s Gh} - \sum_{n=1}^{\infty}\beta_n\left(D_{2n}\sinh\beta_n x - \frac{1}{c^2}D_{3n}\sinh\vartheta_n x\right)\cos\beta_n y,\\
\phi_y^H &= -\frac{\partial w^K}{\partial y} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_y^K}{K_s Gh} + \sum_{n=1}^{\infty}\left(\beta_n D_{2n}\cosh\beta_n x - \frac{\vartheta_n}{c^2}D_{3n}\cosh\vartheta_n x\right)\sin\beta_n y,\\
M_{xx}^H &= M_{xx}^K - \mathscr{B}\frac{\partial Q_y^K}{\partial y} - D(1-\nu)\sum_{n=1}^{\infty}\beta_n^2\left(D_{2n}\cosh\beta_n x - \frac{\mathscr{A}\vartheta_n}{c^2\beta_n}D_{3n}\cosh\vartheta_n x\right)\cos\beta_n y,\\
M_{yy}^H &= M_{yy}^K - \mathscr{B}\frac{\partial Q_x^K}{\partial x} + D(1-\nu)\sum_{n=1}^{\infty}\beta_n^2\left(D_{2n}\cosh\beta_n x - \frac{\mathscr{A}\vartheta_n}{c^2\beta_n}D_{3n}\cosh\vartheta_n x\right)\cos\beta_n y,\\
M_{xy}^H &= M_{xy}^K + \mathscr{B}\frac{\partial Q_y^K}{\partial x} + D(1-\nu)\sum_{n=1}^{\infty}\left[\beta_n^2 D_{2n}\sinh\beta_n x - \frac{\mathscr{A}}{2c^2}(\vartheta_n^2 + \beta_n^2)D_{3n}\sinh\beta_n x\right]\sin\beta_n y,\\
Q_x^H &= Q_x^K + \frac{\mathscr{A}}{2}D(1-\nu)\sum_{n=1}^{\infty}(\beta_n D_{3n}\sinh\vartheta_n x)\cos\beta_n y,\\
Q_y^H &= Q_y^K - \frac{\mathscr{A}}{2}D(1-\nu)\sum_{n=1}^{\infty}(\vartheta_n D_{3n}\cosh\vartheta_n x)\cos\beta_n y.
\end{aligned}
\tag{A.3}
$$

(b) *Anti-symmetrical case.* $M_{xx}|_{x=a/2} = -M_{xx}|_{x=-a/2}$

$$
\begin{aligned}
w^H &= w^K + \left(1 - \frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}^K}{K_s Gh} + \sum_{n=1}^{\infty}(D_{2n}\sinh\beta_n x)\cos\beta_n y,\\
\phi_x^H &= -\frac{\partial w^K}{\partial x} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_x^K}{K_s Gh} - \sum_{n=1}^{\infty}\beta_n\left(D_{2n}\cosh\beta_n x - \frac{1}{c^2}D_{3n}\cosh\vartheta_n x\right)\cos\beta_n y,\\
\phi_y^H &= -\frac{\partial w^K}{\partial y} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_y^K}{K_s Gh} + \sum_{n=1}^{\infty}\left(\beta_n D_{2n}\sinh\beta_n x - \frac{\vartheta_n}{c^2}D_{3n}\sinh\vartheta_n x\right)\sin\beta_n y,\\
M_{xx}^H &= M_{xx}^K - \mathscr{B}\frac{\partial Q_y^K}{\partial y} - D(1-\nu)\sum_{n=1}^{\infty}\beta_n^2\left(D_{2n}\sinh\beta_n x - \frac{\mathscr{A}\vartheta_n}{c^2\beta_n}D_{3n}\sinh\vartheta_n x\right)\cos\beta_n y,\\
M_{yy}^H &= M_{yy}^K - \mathscr{B}\frac{\partial Q_x^K}{\partial x} + D(1-\nu)\sum_{n=1}^{\infty}\beta_n^2\left(D_{2n}\sinh\beta_n x - \frac{\mathscr{A}\vartheta_n}{c^2\beta_n}D_{3n}\sinh\vartheta_n x\right)\cos\beta_n y,\\
M_{xy}^H &= M_{xy}^K + \mathscr{B}\frac{\partial Q_y^K}{\partial x} + D(1-\nu)\sum_{n=1}^{\infty}\left[\beta_n^2 D_{2n}\cosh\beta_n x - \frac{\mathscr{A}}{2c^2}(\vartheta_n^2 + \beta_n^2)D_{3n}\cosh\beta_n x\right]\sin\beta_n y,\\
Q_x^H &= Q_x^K + \frac{\mathscr{A}}{2}D(1-\nu)\sum_{n=1}^{\infty}(\beta_n D_{3n}\cosh\vartheta_n x)\cos\beta_n y,\\
Q_y^H &= Q_y^K - \frac{\mathscr{A}}{2}D(1-\nu)\sum_{n=1}^{\infty}(\vartheta_n D_{3n}\sinh\vartheta_n x)\cos\beta_n y.
\end{aligned}
\tag{A.4}
$$

A.2. Rectangular plates with two opposite simply supported edges

Besides the plate problems (SCSC and SFSF plates) treated in Section 6.2, the following bending relationships can be applied to plate problems whereby the boundary conditions along the other two plate edges can be any combination of free, clamped or simply support:

$$
\begin{aligned}
w^H &= w^K + \left(1 - \frac{\mathscr{B}c^2}{2}\right)\frac{\mathscr{M}^K}{K_s Gh} + \sum_{m=1}^{\infty}\left[\left(C_{3m} - \frac{y}{2\alpha_m}C_{1m}\right)\cosh\alpha_m y \right.\\
&\quad \left. + \left(C_{4m} - \frac{y}{2\alpha_m}C_{2m}\right)\sinh\alpha_m y\right]\sin\alpha_m x,\\
\phi_x^H &= -\frac{\partial w^K}{\partial x} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_x^K}{K_s Gh}\\
&\quad + \sum_{m=1}^{\infty}\left\{\alpha_m\left[\left(\frac{\mathscr{D}}{\mathscr{A}}C_{2m} - C_{3m} + \frac{y}{2\alpha_m}C_{1m}\right)\cosh\alpha_m y \right.\right.\\
&\quad \left. + \left(\frac{\mathscr{D}}{\mathscr{A}}C_{1m} - C_{4m} + \frac{y}{2\alpha_m}C_{2m}\right)\sinh\alpha_m y\right]\\
&\quad \left. + \frac{\lambda_m}{c^2}(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\right\}\cos\alpha_m x,\\
\phi_y^H &= -\frac{\partial w^K}{\partial y} - \left(1 - \frac{1}{\mathscr{A}} - \frac{\mathscr{B}c^2}{2}\right)\frac{Q_y^K}{K_s Gh}\\
&\quad - \sum_{m=1}^{\infty}(\alpha_m)\left[\left(\frac{\mathscr{D}}{A}C_{2m} - C_{3m} + \frac{y}{2\alpha}C_{1m} + \frac{1}{\alpha_m^2}C_{2m}\right)\sinh\alpha_m y \right.\\
&\quad + \left(\frac{\mathscr{D}}{\mathscr{A}}C_{1m} - C_{4m} + \frac{y}{2\alpha}C_{2m} + \frac{1}{\alpha_m^2}C_{1m}\right)\cosh\alpha_m y\\
&\quad \left. + \frac{1}{c^2}(C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\right]\sin\alpha_m x,\\
M_{xx}^H &= M_{xx}^K - \mathscr{B}\frac{\partial Q_y^K}{\partial y} + vD\sum_{m=1}^{\infty}(C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y)\sin\alpha_m x\\
&\quad - D(1-v)\sum_{m=1}^{\infty}(\alpha_m^2)\left[\left(\mathscr{D}C_{2m} - C_{3m} + \frac{y}{2\alpha_m}C_{1m}\right)\cosh\alpha_m y \right.\\
&\quad + \left(\mathscr{D}C_{1m} - C_{4m} + \frac{y}{2\alpha_m}C_{2m}\right)\sinh\alpha_m y\\
&\quad \left. + \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\right]\sin\alpha_m x,\\
M_{yy}^H &= M_{yy}^K - \mathscr{B}\frac{\partial Q_x^K}{\partial x} + D\sum_{m=1}^{\infty}(C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y)\sin\alpha_m x\\
&\quad + D(1-v)\sum_{m=1}^{\infty}(\alpha_m^2)\left[\left(\mathscr{D}C_{2m} - C_{3m} + \frac{y}{2\alpha_m}C_{1m}\right)\cosh\alpha_m y \right.\\
&\quad + \left(\mathscr{D}C_{1m} - C_{4m} + \frac{y}{2\alpha_m}C_{2m}\right)\sinh\alpha_m y
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathscr{A}\lambda_m}{c^2\alpha_m}(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\bigg]\sin\alpha_m x,\\
M_{xy}^H &= M_{xy}^K + \mathscr{B}\frac{\partial Q_y^K}{\partial x} + D(1-v)\sum_{m=1}^{\infty}\bigg\{(\alpha_m^2)\bigg[\bigg(\mathscr{D}C_{2m} - C_{3m} + \frac{y}{2\alpha_m}C_{1m} + \frac{1}{2\alpha_m^2}C_{2m}\bigg)\sinh\alpha_m y\\
&+\bigg(\mathscr{D}C_{1m} - C_{4m} + \frac{y}{2\alpha_m}C_{2m} + \frac{1}{2\alpha_m^2}C_{1m}\bigg)\cosh\alpha_m y\bigg]\\
&+\frac{\mathscr{A}}{2c^2}(\lambda_m^2+\alpha_m^2)(C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\bigg\}\cos\alpha_m x,\\
Q_x^H &= Q_x^K + D\sum_{m=1}^{\infty}(\alpha_m)\bigg[C_{1m}\sinh\alpha_m y + C_{2m}\cosh\alpha_m y\\
&+\frac{\mathscr{A}\lambda_m}{2\alpha_m}(1-v)(C_{5m}\cosh\lambda_m y + C_{6m}\sinh\lambda_m y)\bigg]\cos\alpha_m x,\\
Q_y^H &= Q_y^K + D\sum_{m=1}^{\infty}(\alpha_m)\bigg[C_{1m}\cosh\alpha_m y + C_{2m}\sinh\alpha_m y\\
&+\frac{\mathscr{A}}{2}(1-v)(C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\bigg]\sin\alpha_m x.
\end{aligned}
\tag{A.5}
$$

A.3. Clamped rectangular plates

Note that the displacement and rotation–slope relationships are not provided here since they have appeared in the main text, Eqs. (6.20) and (6.21).

$$
\begin{aligned}
M_{xx}^H &= M_{xx}^K - \mathscr{B}\frac{\partial Q_y^K}{\partial y} + D\bigg[v\sum_{m=1}^{\infty}(C_{1m}\cosh\alpha_m y)\cos\alpha_m x + \sum_{n=1}^{\infty}(D_{1n}\cosh\beta_n x)\cos\beta_n y\bigg]\\
&- D(1-v)\bigg\{\sum_{m=1}^{\infty}(\alpha_m^2)\bigg[\bigg(\frac{\mathscr{D}}{\mathscr{A}}C_{1m} - C_{2m}\bigg)\cosh\alpha_m y + \frac{y}{2\alpha_m}C_{1m}\sinh\alpha_m y\\
&- \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}C_{3m}\cosh\lambda_m y\bigg]\cos\alpha_m x - \sum_{n=1}^{\infty}(\beta_n^2)\bigg[\bigg(\frac{\mathscr{D}}{\mathscr{A}}D_{1n} - D_{2n}\bigg)\cosh\beta_n x\\
&+ \frac{x}{2\beta_n}D_{1n}\sinh\beta_n x + \frac{\mathscr{A}\vartheta_n}{c^2\beta_n}D_{3n}\cosh\vartheta_n x\bigg]\cos\beta_n y\bigg\},\\
M_{yy}^H &= M_{yy}^K - \mathscr{B}\frac{\partial Q_x^K}{\partial x} + D\bigg[\sum_{m=1}^{\infty}(C_{1m}\cosh\alpha_m y)\cos\alpha_m x + v\sum_{n=1}^{\infty}(D_{1n}\cosh\beta_n x)\cos\beta_n y\bigg]\\
&+ D(1-v)\bigg\{\sum_{m=1}^{\infty}(\alpha_m^2)\bigg[\bigg(\frac{\mathscr{D}}{\mathscr{A}}C_{1m} - C_{2m}\bigg)\cosh\alpha_m y + \frac{y}{2\alpha_m}C_{1m}\sinh\alpha_m y\\
&- \frac{\mathscr{A}\lambda_m}{c^2\alpha_m}C_{3m}\cosh\lambda_m y\bigg]\cos\alpha_m x
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sum_{n=1}^{\infty} (\beta_n^2) \left[\left(\frac{\mathscr{D}}{\mathscr{A}} D_{1n} - D_{2n} \right) \cosh \beta_n x + \frac{x}{2\beta_n} D_{1n} \sinh \beta_n x + \frac{\mathscr{A} \vartheta_n}{c^2 \beta_n} D_{3n} \cosh \vartheta_n x \right] \cos \beta_n y \Bigg\}, \\
M_{xy}^H &= M_{xy}^K + \mathscr{B} \frac{\partial Q_y^K}{\partial x} - D(1-v) \left(\sum_{m=1}^{\infty} \left\{ (\alpha_m^2) \left[\left(\frac{\mathscr{D}}{\mathscr{A}} C_{1m} - C_{2m} + \frac{1}{2\alpha_m^2} C_{1m} \right) \sinh \alpha_m y \right. \right. \right. \\
&\quad \left. \left. + \frac{y}{2\alpha_m} C_{1m} \cosh \alpha_m y \right] - \frac{\mathscr{A}}{2c^2} (\lambda_m^2 + \alpha_m^2) C_{3m} \sinh \lambda_m y \right\} \sin \alpha_m x \\
&\quad + \sum_{n=1}^{\infty} \left\{ (\beta_n^2) \left[\left(\frac{\mathscr{D}}{\mathscr{A}} D_{1n} - D_{2n} + \frac{1}{2\beta_n^2} D_{1n} \right) \sinh \beta_n x + \frac{x}{2\beta_n} D_{1n} \cosh \beta_n x \right] \right. \\
&\quad \left. \left. + \frac{\mathscr{A}}{2c^2} (\vartheta_n^2 + \beta_n^2) D_{3n} \sinh \vartheta_n x \right\} \sin \beta_n y \right), \\
Q_x^H &= Q_x^K + D \left\{ \sum_{m=1}^{\infty} \left[\frac{\mathscr{A}}{2} \lambda_m (1-v) C_{3m} \cosh \lambda_m y - \alpha_m C_{1m} \cosh \alpha_m y \right] \sin \alpha_m x \right. \\
&\quad \left. + \sum_{n=1}^{\infty} (\beta_n) \left[\frac{\mathscr{A}}{2} (1-v) D_{3n} \sinh \vartheta_n x + D_{1n} \sinh \beta_n x \right] \cos \beta_n y \right\}, \\
Q_y^H &= Q_y^K + D \left\{ \sum_{m=1}^{\infty} (\alpha_m) \left[C_{1m} \sinh \alpha_m y - \frac{\mathscr{A}}{2} (1-v) C_{3m} \sinh \lambda_m y \right] \cos \alpha_m x \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \left[\frac{\mathscr{A}}{2} \vartheta_n (1-v) D_{3n} \cosh \vartheta_n x + \beta_n D_{1n} \cosh \beta_n x \right] \sin \beta_n y \right\}.
\end{aligned}
\tag{A.6}
$$

Appendix B. Classical thin plate solutions

In this appendix, the classical plate solutions are provided for one to apply readily to bending relationships as derived in the earlier section and together with those specialized in Appendix A. The solutions are furnished for the various plate problems considered wherever available.

B.1. Simply supported rectangular plates

B.1.1. Distributed moments along $y = \pm b/2$

In general, the distributed moment applied can be represented in Fourier series as (Timoshenko and Woinowsky-Krieger, 1959)

$$
M_0(x) = \sum_{m=1}^{\infty} E_m \sin \frac{m\pi}{2} \cos \alpha_m x, \tag{B.1}
$$

where $E_m = 4M_0/m\pi$ for the particular case of uniformly distributed moment $M_0(x) = M_0$.

(a) *Symmetrical case.* $M_{yy}^K|_{y=b/2} = M_{yy}^K|_{y=-b/2} = M_0(x)$

Consider such a symmetry and Eq. (B.1), one can write the deflection and stress resultants as

$$\begin{aligned}
w^K(x,y) &= \frac{1}{2D}\sum_{m=1}^{\infty}\operatorname{sech}\frac{\alpha_m b}{2}\left(\frac{E_m}{\alpha_m^2}\right)\left(\frac{\alpha_m b}{2}\tanh\frac{\alpha_m b}{2}\cosh\alpha_m y\right.\\
&\qquad\left. -\alpha_m y\sinh\alpha_m y\right)\sin\frac{m\pi}{2}\cos\alpha_m x,\\
M_{xx}^K(x,y) &= \frac{1}{2}\sum_{m=1}^{\infty}E_m\operatorname{sech}\frac{\alpha_m b}{2}\left\{\left[2v+(1-v)\frac{\alpha_m b}{2}\tanh\frac{\alpha_m b}{2}\right]\cosh\alpha_m y\right.\\
&\qquad\left. -(1-v)\alpha_m y\sinh\alpha_m y\right\}\sin\frac{m\pi}{2}\cos\alpha_m x,\\
M_{yy}^K(x,y) &= \frac{1}{2}\sum_{m=1}^{\infty}E_m\operatorname{sech}\frac{\alpha_m b}{2}\left\{\left[2-(1-v)\frac{\alpha_m b}{2}\tanh\frac{\alpha_m b}{2}\right]\cosh\alpha_m y\right.\\
&\qquad\left. +(1-v)\alpha_m y\sinh\alpha_m y\right\}\sin\frac{m\pi}{2}\cos\alpha_m x,\\
\mathscr{M}^K(x,y) &= \sum_{m=1}^{\infty}\left(E_m\operatorname{sech}\frac{\alpha_m b}{2}\cosh\alpha_m y\right)\sin\frac{m\pi}{2}\cos\alpha_m x,\\
M_{xy}^K(x,y) &= \frac{1}{2}(1-v)\sum_{m=1}^{\infty}E_m\operatorname{sech}\frac{\alpha_m b}{2}\left[\left(\frac{\alpha_m b}{2}\tanh\frac{\alpha_m b}{2}-1\right)\sinh\alpha_m y\right.\\
&\qquad\left. -\alpha_m y\cosh\alpha_m y\right]\sin\frac{m\pi}{2}\sin\alpha_m x,\\
Q_x^K(x,y) &= -\sum_{m=1}^{\infty}\left(\alpha_m E_m\operatorname{sech}\frac{\alpha_m b}{2}\cosh\alpha_m y\right)\sin\frac{m\pi}{2}\sin\alpha_m x,\\
Q_y^K(x,y) &= \sum_{m=1}^{\infty}\left(\alpha_m E_m\operatorname{sech}\frac{\alpha_m b}{2}\sinh\alpha_m y\right)\sin\frac{m\pi}{2}\cos\alpha_m x.
\end{aligned}\tag{B.2}$$

(b) *Anti-symmetrical case.* $M_{yy}^K|_{y=b/2} = -M_{yy}^K|_{y=-b/2} = M_0(x)$

For this case, the deflection and stress resultants can be derived as

$$\begin{aligned}
w^K(x,y) &= \frac{1}{2D}\sum_{m=1}^{\infty}\operatorname{cosech}\frac{\alpha_m b}{2}\left(\frac{E_m}{\alpha_m^2}\right)\left(\frac{\alpha_m b}{2}\coth\frac{\alpha_m b}{2}\sinh\alpha_m y\right.\\
&\qquad\left. -\alpha_m y\cosh\alpha_m y\right)\sin\frac{m\pi}{2}\cos\alpha_m x,\\
M_{xx}^K(x,y) &= \frac{1}{2}\sum_{m=1}^{\infty}E_m\operatorname{cosech}\frac{\alpha_m b}{2}\left\{\left[2v+(1-v)\frac{\alpha_m b}{2}\coth\frac{\alpha_m b}{2}\right]\sinh\alpha_m y\right.\\
&\qquad\left. -(1-v)\alpha_m y\cosh\alpha_m y\right\}\sin\frac{m\pi}{2}\cos\alpha_m x,\\
M_{yy}^K(x,y) &= \frac{1}{2}\sum_{m=1}^{\infty}E_m\operatorname{cosech}\frac{\alpha_m b}{2}\left\{\left[2-(1-v)\frac{\alpha_m b}{2}\coth\frac{\alpha_m b}{2}\right]\sinh\alpha_m y\right.\\
&\qquad\left. +(1-v)\alpha_m y\cosh\alpha_m y\right\}\sin\frac{m\pi}{2}\cos\alpha_m x,\\
\mathscr{M}^K(x,y) &= \sum_{m=1}^{\infty}\left(E_m\operatorname{cosech}\frac{\alpha_m b}{2}\sinh\alpha_m y\right)\sin\frac{m\pi}{2}\cos\alpha_m x,
\end{aligned}$$

$$
\begin{aligned}
M_{xy}^K(x,y) &= \frac{1}{2}(1-v)\sum_{m=1}^{\infty} E_m \operatorname{cosech}\frac{\alpha_m b}{2}\left[\left(\frac{\alpha_m b}{2}\coth\frac{\alpha_m b}{2}-1\right)\cosh\alpha_m y\right.\\
&\quad \left. -\,\alpha_m y\sinh\alpha_m y\right]\sin\frac{m\pi}{2}\sin\alpha_m x,\\
Q_x^K(x,y) &= -\sum_{m=1}^{\infty}\left(\alpha_m E_m \operatorname{cosech}\frac{\alpha_m b}{2}\sinh\alpha_m y\right)\sin\frac{m\pi}{2}\sin\alpha_m x,\\
Q_y^K(x,y) &= \sum_{m=1}^{\infty}\left(\alpha_m E_m \operatorname{cosech}\frac{\alpha_m b}{2}\cosh\alpha_m y\right)\sin\frac{m\pi}{2}\cos\alpha_m x.
\end{aligned}
\tag{B.3}
$$

B.1.2. Distributed moments along $x = \pm a/2$

Like the previous plate problem, one can express the distributed moment applied using Fourier series as (Timoshenko and Woinowsky-Krieger, 1959)

$$
M_0(y) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi}{2}\cos\beta_n y,
\tag{B.4}
$$

where $F_n = 4M_0/n\pi$ for the case of uniformly distributed moment $M_0(y) = M_0$.

(a) *Symmetrical case.* $M_{xx}^K|_{x=a/2} = M_{xx}^K|_{x=-a/2} = M_0(y)$

In view of the loading symmetry and Eq. (B.4), the thin plate deflection and stress resultants are given as

$$
\begin{aligned}
w^K(x,y) &= \frac{1}{2D}\sum_{n=1}^{\infty}\operatorname{sech}\frac{\beta_n a}{2}\left(\frac{F_n}{\beta_n^2}\right)\left(\frac{\beta_n a}{2}\tanh\frac{\beta_n a}{2}\cosh\beta_n x\right.\\
&\quad \left. -\,\beta_n x\sinh\beta_n x\right)\sin\frac{n\pi}{2}\cos\beta_n y,\\
M_{xx}^K(x,y) &= \frac{1}{2}\sum_{n=1}^{\infty} F_n \operatorname{sech}\frac{\beta_n a}{2}\left\{\left[2-(1-v)\frac{\beta_n a}{2}\tanh\frac{\beta_n a}{2}\right]\cosh\beta_n x\right.\\
&\quad \left. +\,(1-v)\beta_n x\sinh\beta_n x\right\}\sin\frac{n\pi}{2}\cos\beta_n y,\\
M_{yy}^K(x,y) &= \frac{1}{2}\sum_{n=1}^{\infty} F_n \operatorname{sech}\frac{\beta_n a}{2}\left\{\left[2v+(1-v)\frac{\beta_n a}{2}\tanh\frac{\beta_n ab}{2}\right]\cosh\beta_n y\right.\\
&\quad \left. -\,(1-v)\beta_n x\sinh\beta_n y\right\}\sin\frac{n\pi}{2}\cos\beta_n y,\\
\mathscr{M}^K(x,y) &= \sum_{n=1}^{\infty}\left(F_n \operatorname{sech}\frac{\beta_n a}{2}\cosh\beta_n x\right)\sin\frac{n\pi}{2}\cos\beta_n y,\\
M_{xy}^K(x,y) &= \frac{1}{2}(1-v)\sum_{n=1}^{\infty} F_n \operatorname{sech}\frac{\beta_n a}{2}\left[\left(\frac{\beta_n a}{2}\tanh\frac{\beta_n a}{2}-1\right)\sinh\beta_n x\right.\\
&\quad \left. -\,\beta_n x\cosh\beta_n x\right]\sin\frac{n\pi}{2}\sin\beta_n y,\\
Q_x^K(x,y) &= \sum_{n=1}^{\infty}\left(\beta_n F_n \operatorname{sech}\frac{\beta_n a}{2}\sinh\beta_n x\right)\sin\frac{n\pi}{2}\cos\beta_n y,\\
Q_y^K(x,y) &= -\sum_{n=1}^{\infty}\left(\beta_n F_n \operatorname{sech}\frac{\beta_n a}{2}\cosh\beta_n x\right)\sin\frac{n\pi}{2}\sin\beta_n y.
\end{aligned}
\tag{B.5}
$$

(b) *Anti-symmetrical case.* $M_{xx}^K|_{x=a/2} = -M_{xx}^K|_{x=-a/2} = M_0(y)$

For the anti-symmetrical case, the deflection and stress resultants can be furnished as

$$
\begin{aligned}
w^K(x,y) &= \frac{1}{2D}\sum_{n=1}^{\infty}\operatorname{cosech}\frac{\beta_n a}{2}\left(\frac{F_n}{\beta_n^2}\right)\left(\frac{\beta_n a}{2}\coth\frac{\beta_n a}{2}\sinh\beta_n x\right.\\
&\quad\left. - \beta_n x\cosh\beta_n x\right)\sin\frac{n\pi}{2}\cos\beta_n y,\\
M_{xx}^K(x,y) &= \frac{1}{2}\sum_{n=1}^{\infty}F_n\operatorname{cosech}\frac{\beta_n a}{2}\left\{\left[2-(1-v)\frac{\beta_n a}{2}\coth\frac{\beta_n a}{2}\right]\sinh\beta_n x\right.\\
&\quad\left. + (1-v)\beta_n x\cosh\beta_n x\right\}\sin\frac{n\pi}{2}\cos\beta_n y,\\
M_{yy}^K(x,y) &= \frac{1}{2}\sum_{n=1}^{\infty}F_n\operatorname{cosech}\frac{\beta_n a}{2}\left\{\left[2v+(1-v)\frac{\beta_n a}{2}\coth\frac{\beta_n ab}{2}\right]\sinh\beta_n y\right.\\
&\quad\left. - (1-v)\beta_n x\cosh\beta_n y\right\}\sin\frac{n\pi}{2}\cos\beta_n y,\\
\mathscr{M}^K(x,y) &= \sum_{n=1}^{\infty}\left(F_n\operatorname{cosech}\frac{\beta_n a}{2}\sinh\beta_n x\right)\sin\frac{n\pi}{2}\cos\beta_n y,\\
M_{xy}^K(x,y) &= \frac{1}{2}(1-v)\sum_{n=1}^{\infty}F_n\operatorname{cosech}\frac{\beta_n a}{2}\left[\left(\frac{\beta_n a}{2}\coth\frac{\beta_n a}{2}-1\right)\cosh\beta_n x\right.\\
&\quad\left. - \beta_n x\sinh\beta_n x\right]\sin\frac{n\pi}{2}\sin\beta_n y,\\
Q_x^K(x,y) &= \sum_{n=1}^{\infty}\left(\beta_n F_n\operatorname{cosech}\frac{\beta_n a}{2}\cosh\beta_n x\right)\sin\frac{n\pi}{2}\cos\beta_n y,\\
Q_y^K(x,y) &= -\sum_{n=1}^{\infty}\left(\beta_n F_n\operatorname{cosech}\frac{\beta_n a}{2}\sinh\beta_n x\right)\sin\frac{n\pi}{2}\sin\beta_n y.
\end{aligned}
\tag{B.6}
$$

B.2. Rectangular plates with two opposite simply supported edges

The thin plate solutions for rectangular plates with two simply supported edges have been well-documented in many standard texts like Timoshenko and Woinowsky-Krieger (1959), Mansfield (1989) and Reddy (1999) and they are presented herein for completeness. Much in the same fashion with the applied edge moments, the applied transverse loading for the considered plate problem can be represented in the Fourier sine series so long the shape of the loading function remains the same along every section perpendicular to the two opposite edges. The transverse loading can be generally written as

$$
q_0(x,y) = \sum_{m=1}^{\infty} q_m(y)\sin\alpha_m x, \tag{B.7}
$$

where $q_m(y) = 4q_0/m\pi$ for the case of uniformly distributed load $q_0(x,y) = q_0$. With the transverse loading given in such form, the deflection and stress resultants can be established as

$$
\begin{aligned}
w^K(x,y) &= \frac{1}{D}\sum_{m=1}^{\infty}\frac{q_m(y)}{\alpha_m^4}(1 + A_m\cosh\alpha_m y + B_m\alpha_m y\sinh\alpha_m y)\sin\alpha_m x,\\
M_{xx}^K(x,y) &= -\sum_{m=1}^{\infty}\frac{q_m(y)}{\alpha_m^2}\{-1 + [2vB_m - (1-v)A_m]\cosh\alpha_m y - (1-v)B_m\alpha_m y\sinh\alpha_m y\}\sin\alpha_m x,\\
M_{yy}^K(x,y) &= -\sum_{m=1}^{\infty}\frac{q_m(y)}{\alpha_m^2}\{-v + [2B_m + (1-v)A_m]\cosh\alpha_m y + (1-v)B_m\alpha_m y\sinh\alpha_m y\}\sin\alpha_m x,\\
\mathscr{M}^K(x,y) &= -\sum_{m=1}^{\infty}\frac{q_m(y)}{\alpha_m^2}(-1 + 2B_m\cosh\alpha_m y)\sin\alpha_m x, \qquad (\text{B}.8)\\
M_{xy}^K(x,y) &= -(1-v)\sum_{n=1}^{\infty}\frac{q_m(y)}{\alpha_m^2}[(A_m + B_m)\sinh\alpha_m y + B_m\alpha_m y\cosh\alpha_m y]\cos\alpha_m x,\\
Q_x^K(x,y) &= -\sum_{m=1}^{\infty}\frac{q_m(y)}{\alpha_m}(1 + 2B_m\cosh\alpha_m y)\cos\alpha_m x,\\
Q_y^K(x,y) &= -\sum_{m=1}^{\infty}\frac{q_m(y)}{\alpha_m}(2B_m\sinh\alpha_m y)\sin\alpha_m x,
\end{aligned}
$$

where

$$
\begin{aligned}
&\textbf{SSSS plates}:\quad A_m = -\frac{2 + \frac{\alpha_m b}{2}\tanh\frac{\alpha_m b}{2}}{2\cosh\frac{\alpha_m b}{2}},\quad B_m = \frac{1}{2\cosh\frac{\alpha_m b}{2}},\\
&\textbf{SCSC plates}:\quad A_m = -\frac{1 + \frac{\alpha_m b}{2}\coth\frac{\alpha_m b}{2}}{\cosh\frac{\alpha_m b}{2} + \frac{\alpha_m b}{2}\operatorname{cosech}\frac{\alpha_m b}{2}},\\
&\qquad\qquad\qquad B_m = \frac{1}{\cosh\frac{\alpha_m b}{2} + \frac{\alpha_m b}{2}\operatorname{cosech}\frac{\alpha_m b}{2}}, \qquad (\text{B}.9)\\
&\textbf{SFSF plates}:\quad A_m = \frac{v(1+v)\sinh\frac{\alpha_m b}{2} - v(1-v)\frac{\alpha_m b}{2}\cosh\frac{\alpha_m b}{2}}{(3+v)(1-v)\sinh\frac{\alpha_m b}{2}\cosh\frac{\alpha_m b}{2} - (1-v)^2\frac{\alpha_m b}{2}},\\
&\qquad\qquad\qquad B_m = \frac{v(1-v)\sinh\frac{\alpha_m b}{2}}{(3+v)(1-v)\sinh\frac{\alpha_m b}{2}\cosh\frac{\alpha_m b}{2} - (1-v)^2\frac{\alpha_m b}{2}}.
\end{aligned}
$$

Note that the solutions presented apply to the coordinate system given in Fig. 2 and one can transform the coordinate system to that as shown in Fig. 3 by substituting $x = \bar{x} + a/2$.

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